PART II

THE HIGH SCHOOL ALGEBRA

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PART II.

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PREFACE.

THE favorable reception accorded to "The High School Algebra, Part I.," by the Mathematical Masters of the leading Collegiate Institutes and High Schools of Ontario, has induced the authors to proceed with Part II., which is now given to the public. Its leading features are similar to those of the former volume. Parts of the subject which are usually treated in a superficial manner, or wholly omitted, have been given considerable prominence. The difficulties of the subject are presented one at a time, in logical order, preceded, where experience has shown it necessary, by numerical illustrations to prepare the way for more general investigations. Explanatory matter and formal proofs of propositions have been kept distinct, as far as possible, for the convenience of students preparing for written examinations. The more important theorems, which should be read by all students, are given in italics; the remainder might be omitted by junior readers and those who are not candidates for Honors. Originality has not been attempted; yet new views of old theorems will be found in many instances, and new theorems, also, in a few cases. Arts. 109, 126 and 152 will probably be found interesting and instructive.

The examples, which are very numerous and varied in their character, have all been tested in the class room, and proved to be suitable before being inserted. Their number is greater than the majority of students will find time for working; but as they are carefully graded it will be easy to select as many as may be desired of any required degree of difficulty. During the first reading of the subject an intelligent solution of from one half to two-thirds of the examples will be amply sufficient.

An effort has been made, by means of diagrams and familiar

illustrations, to show clearly the connection between the symbols on paper and the actual quantities they represent. This will be especially noticed in the chapter on Imaginary Quantities, which has been treated wholly from a geometrical point of view. Experience seems to warrant the belief that this method will prove interesting and instructive to the student who limits his attention to ordinary Algebra; whilst to those who pursue their way through the higher mathematics it will serve as an introduction to the new and beautiful science of Quaternions.

Throughout the whole work the authors have constantly kept in mind the future as well as the immediate wants of the student. The treatment of Homogeneous Equations will be found convenient in Conic Sections; the Theory of Infinite Quantities prepares the way for the Calculus; whilst it will afterwards be seen that many of the examples give the solution of problems in various departments of more advanced work.

The materials used in the preparation of the present work have been gathered from many sources. The standard Algebras of Wood, Potts and Todhunter have furnished a considerable portion. The more recent works of Chrystal, C. Smith, Whitworth, Hall & Knight, Newcomb, Wentworth, and Ch. De Comberousse, have also been consulted. The papers set at the various University and Departmental Examinations have furnished many examples, whilst many more have been constructed for the use of the authors' own classes in the regular course of instruction. The authors have also to thank Prof. Alfred Baker, of University College, for reading the proof sheets of the chapter on Imaginary Quantities, and for valuable criticisms and suggestions on that subject.

Should the present work be received with sufficient favor it will be followed by a third volume, treating of the remaining portions of the subject, so far as it is usually read for the B.A. Degree with Honors in any Canadian or American University.

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HIGHER ALGEBRA.

CHAPTER I.

FUNDAMENTAL CONCEPTIONS AND OPERATIONS.

ON THE STUDY OF ALGEBRA.

1. The Science of Algebra has for its object the investigation of the magnitude and relations existing between the various quantities which are capable of being represented by numbers. The process of investigation is carried on by means of symbols representing the quantities and the relations they bear to each other. These symbols may be divided into two great classes, viz.:

1. Symbols of Quantity and Relation.

2. Symbols of Operation.

In the application of Algebra to the solution of any practical problem there are four distinct stages necessary, as follows:

1. A clear conception of the nature and relations of the quantities involved.

2. An accurate representation of those quantities and their relations in algebraic symbols.

3. A proper performance with those symbols of the operations demanded by the conditions of the problem.

4. A correct interpretation of the result of the symbolical operations.

The truth of the information thus derived in any particular case depends upon accuracy in each of these four stages. It is therefore of the greatest importance for the student to clearly perceive the connection between the symbols on paper and the quantities or operations which they represent.

2. It should be carefully observed that the symbols of Algebra, in common with all symbols, have no meaning in themselves, but only such meaning as may be attached to them by common consent of those who use them. In determining the meaning to be assigned in any particular case, we may proceed by either of two distinct methods. We may first assign an arbitrary meaning to a symbol of quantity, and then determine the laws of the operations to which it may be subjected; or first assume that it obeys certain laws, and then assign the meaning which will enable it to do so.

The former is the Synthetic method, and is the most suitable for demonstrations and for conveying information; the latter is the Analytic method, and the one by which discoveries are made and the boundaries of the science enlarged. Examples of each method are found in Part I., positive and negative quantities are treated synthetically, but Indices analytically. Further examples of each method will also be given in the present work.

3. When two or more operations are to be performed successively, care must be taken to observe the proper order in performing them. Certain operations are interchangeable, others are not; no change must be made without examining whether such change will affect the result. The following are the fundamental laws of elementary Algebra on this point:

I. THE LAW OF COMMUTATION.

1. *Additions and subtractions may be performed in any order.*

Thus $\qquad a+b-c=a-c+b=b-c+a=-c+a+b.$

2. *Multiplications and divisions may be performed in any order.*

Thus
$$a \times b \times c = b \times c \times a = c \times a \times b;$$
$$(a \times b) \div c = a \times (b \div c) = (a \div c) \times b.$$

3. *Involutions and evolutions may be performed in any order; they are also interchangeable with multiplications and divisions.*

Thus
$$(a^m)^n = (a^n)^m; \quad (a^{\frac{1}{m}})^{\frac{1}{n}} = (a^{\frac{1}{n}})^{\frac{1}{m}}; \quad (a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m;$$
$$a^m b^m = (ab)^m; \quad a^{\frac{1}{m}} b^{\frac{1}{m}} = (ab)^{\frac{1}{m}}; \quad \frac{a^{\frac{1}{m}}}{b^{\frac{1}{m}}} = \left(\frac{a}{b}\right)^{\frac{1}{m}}$$

II. THE LAW OF DISTRIBUTION.

1. *Additions and subtractions of numbers may be distributed over a series of additions and subtractions of their parts.*

Thus
$$a + (b + c - d) = a + b + c - d;$$
$$a - (b + c - d) = a - b - c + d.$$

2. *Multiplications (and divisions) of numbers by one another may be distributed over a series of additions and subtractions of the products (and quotients) of their parts.*

Thus
$$(a - b + c)m = am - bm + cm;$$
$$(a - b)(c - d) = (a - b)c - (a - b)d$$
$$= ac - bc - ad + bd;$$
$$(a + b - c) \div m = a \div m + b \div m - c \div m.$$

4. The exact meaning of the examples in the preceding Art. should be carefully noted, expressed in words, and, where possible, illustrated by concrete quantities. Thus in the third example under the Law of Distribution the first combination of symbols directs us to subtract b from a, add c to the difference, and multiply the sum by m. The second combination requires the multiplication of the several parts to be performed first, and the additions and subtractions to be performed afterwards. The greater

part of algebraic work consists in such interchange of operations, and the mistakes of beginners are usually due to an improper application of these fundamental laws.

The remainder of this chapter is devoted to an explanation of the meaning attached to various symbols of quantity and operations which do not occur in elementary work. The student must not expect to be able to grasp the full meaning of some portions of it at the first reading, but frequent reference to it in connection with the following chapters will be found helpful and instructive.

INFINITE QUANTITIES.

5. Quantity has already been defined to be that which is capable of being divided into parts; it remains to distinguish **Finite** and **Infinite Quantities.**

6. A **Finite Quantity** is one which has boundaries or limits; it is a definite portion of any magnitude.

All Quantities treated of in ordinary arithmetical or algebraical operations are finite.

7. An **Infinite Quantity** is one which has no boundaries or limits; it is magnitude considered without limitation.

Time and Space, taken in their general signification, are familiar examples of infinite Quantity. We cannot conceive of any limit to the duration of Time or to the extent of Space; they are therefore said to be infinite.

SYMBOLS OF INFINITY.

8. The series of numbers, 1, 2, 3, 4, etc., may evidently be continued without limit. No particular number is so great that it cannot be doubled and thus rendered greater. Number, then, like Time and Space, having no limit, is infinite.

The symbol ∞ is frequently used to denote "infinity." Its precise meaning cannot be made evident by a single definition, but it will be clearly illustrated in the following Arts.

9. The series of negative numbers, -1, -2, -3, etc., may also be continued to any extent; and the two series thus form a system extending to nfinity in each direction from zero as the central or starting point, the *number* in each case denoting *distance*, and the *sign*, *direction*.

10. When magnitude only is to be considered it is frequently convenient to consider another infinite series, in which the central number is a unit or 1; and as the series 2, 3, 4, etc., is derived from it by *multiplication*, so another series, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, etc., may be derived from it by *division*. The one series becomes indefinitely great as before, but the other becomes indefinitely small. We place ∞ at the end of one series to show that it is to be endlessly increased, and 0 at the end of the other to show that it is to be endlessly diminished. The whole series in order will be

$$\frac{1}{\infty} \text{ or } 0 \ldots . \frac{1}{3}, \frac{1}{2}, 1, 2, 3 \ldots . \infty .$$

It should he carefully observed that the above series is infinite in each direction from the unit; and as the infinitely great or ∞ can never be really attained, neither can the infinitely small or 0 ever be reached. Absolute zero can be obtained only by subtraction, never by division. This is giving a new meaning to the symbol 0, which formerly denoted the mere absence of quantity. There will be no difficulty in determining the meaning intended in any particular instance; and it will be found that this extended meaning adds immensely to the power of Algebra as an instrument of investigation.

OPERATIONS WITH SYMBOLS OF INFINITY.

11. Since 0 and ∞ do not represent any definite numbers or quantities, operations performed with these symbols are not subject in all cases to the laws governing the same operations with symbols representing definite or finite quantities, but require a separate investigation.

12. *A finite quantity divided by 0 gives ∞ for quotient.*

Let $\frac{a}{x}=q$, then $a=qx$. Now let x continually diminish and q consequently continually increase, so that their product is always equal to the constant quantity a; it is evident that by making x small enough q may be made greater than any assignable finite quantity. This is briefly expressed in symbols thus, $\frac{a}{0}=\infty$.

13. *A finite quantity divided by ∞ gives 0 for quotient.*

Let $\frac{a}{x}=q$, then $a=qx$. Now let x continually increase and q consequently continually diminish; then, by making x great enough, q can be made less than any assignable finite quantity, or in symbols $\frac{a}{\infty}=0$.

14. *Zero divided by zero or infinity by infinity gives any quantity whatever for quotient.*

Let $\frac{a}{x}=q$, then $a=qx$. Now, if a and x are each zero, q may be any finite quantity whatever. The same is also true if a and x are each ∞, since q times 0 or ∞ is 0 or ∞.

NOTE—If the *origin* of the symbols 0 or ∞ be known, then a definite meaning can usually be attached to the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$, as will be shown further on.

15. *If $a > 1$, $a^\infty=\infty$; and if $a < 1$, $a^\infty=0$.*

The truth of this proposition will be easily perceived by considering a simple example.

Let $a=\frac{4}{3}$, then, by taking the 2nd, 3rd, 4th, etc., powers in succession, it will be observed that each multiplication adds more than $\frac{1}{3}$ to the original fraction; therefore by multiplying a suffi-

cient number of times the result may be made greater than any finite quantity, or $a^{\infty} = \infty$.

Again, if $a < 1$, let $a = \frac{1}{p}$; then $p > 1$ and $a^{\infty} = \frac{1}{p^{\infty}} = \frac{1}{\infty} = 0$.

The truth of this latter proposition is generally assumed by merely noting that if a is less than unity, each term of the series a, a^2, a^3, etc., is less than the preceding, and therefore by taking the exponent large enough the result may be made less than any finite quantity. But this mode of reasoning is fallacious, for the terms of the series $\frac{1}{2}$, $\left(\frac{2}{3}\right)^2$, $\left(\frac{3}{4}\right)^3$, $\left(\frac{4}{5}\right)^4$, etc., continually decrease, and yet if the series be continued to any extent the terms will always be greater than $\frac{1}{3}$.

16. Fractions which take the form $\frac{0}{0}$ when particular values are given to the literal symbols involved are termed **Vanishing Fractions.** They usually arise from the numerator and denominator having a common factor, which is zero for the given values. Such fractions have no definite value if by 0 we mean the entire absence of quantity; but with the meaning assigned in Art. 10 a definite value may generally be found.

Ex. 1.—To find the limit of the value of $\frac{x^2 - a^2}{x - a}$ when the value of x approaches the value of a.

Let $x = a + h$, then $\frac{x^2 - a^2}{x - a} = \frac{(a+h)^2 - a^2}{(a+h) - a} = \frac{2ah + h^2}{h} = 2a + h$. Now let the value of h become indefinitely small, then the value of the fraction, viz., $2a + h$, becomes indefinitely near to $2a$, *i.e.*, by making x sufficiently near in value to a, the value of the fraction may be made as nearly equal as we please to $2a$, the limit required.

Practically this result is found at once by removing the common factor $x - a$ and writing a for x in the quotient.

Ex. 2.—To find the value of $\frac{2x^2+3x+1}{3x^2+7x+4}$ when $x=-1$ and when $x=\infty$.

When $x=-1$ both numerator and denominator vanish, therefore $x+1$ is a factor. Removing this factor we get $\frac{2x+1}{3x+4}$; substituting -1 for x we get -1, the result required.

When x is infinitely great both terms of the fraction become ∞; its value, therefore, in this form is indeterminate. The fraction, however, in its lowest terms may be written $\frac{2+\frac{1}{x}}{3+\frac{4}{x}}$; and if we now put $x=\infty$, $\frac{1}{x}$ and $\frac{4}{x}$ each $=0$, and the fraction becomes $\frac{2}{3}$. The meaning in this case is, "By making x sufficiently great the value of the fraction may be made as nearly equal as we please to $\frac{2}{3}$."

17. The product of two factors vanishes when one factor vanishes, providing the other remains finite; if the second factor becomes infinite the product may be zero, finite or infinite, as the following simple examples show:

Let $x=a^2$, and let (1) $y=\frac{1}{a}$, (2) $y=\frac{1}{a^2}$, (3) $y=\frac{1}{a^3}$; then in each case when $x=0$, $a=0$ and $y=\infty$.

Thus,
$$(1)\ xy=a^2.\frac{1}{a}=a=0.$$
$$(2)\ xy=a^2.\frac{1}{a^2}=1.$$
$$(3)\ xy=a^2.\frac{1}{a^3}=\frac{1}{a}=\infty.$$

In the above and all similar examples the meaning assigned to 0 is that of Art. 10; with the ordinary meaning the above process would be wholly unintelligible. It should be further observed the two factors become, the one 0, the other ∞, by the vanishing

of the same quantity, a; otherwise no definite result could be given. For example, if $x=a$ and $y=\frac{1}{b}$, and if a and b each become 0, then $xy=a\times\frac{1}{b}=0\times\infty$; but this product is entirely indefinite.

EXERCISE I.

1. Find the value of $\frac{x^n-a^n}{x-a}$ when $x=a$, n being a positive integer.

2. Find the value of $\frac{x^n+a^n}{x+a}$ when $x=-a$, (1) if n be odd; (2) if n be even.

3. Find the value of $\frac{1-3x^2+2x^3}{(1-x)^2}$ when $x=1$.

4. Find the value of $\frac{x^2+5x+8}{x^3+3x^2-4x}$ when $x=0$ and when $x=\infty$.

5. If $x=1$, find the value of $\frac{x^3-3x+2}{(x-1)^n}$ when $n=1$, 2 and 3.

6. Find the value of $\frac{(a^2-x^2)^{\frac{1}{2}}+(a-x)^{\frac{3}{2}}}{(a^3-x^3)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}$ when $x=a$.

7. Find the value of $\frac{\sqrt{x}-\sqrt{2a}+\sqrt{x-2a}}{\sqrt{x^2-4a^2}}$ when $x=2a$.

8. Find the value of $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ when $a=0$.

9. Find the value of $\frac{a}{1-a^{\frac{1}{x}}}$ when $x=0$ and when $x=\infty$.

10. Find the value of $\frac{x^3-a^3}{y^3-b^3}$ when $x=a$ and $y=b$.

11. If $x^2+y^2-(2y+a-b)x+(a-b)y=ab$, find the value of $\frac{x^2-(3a-b)x+2a(a-b)}{y^3-(a^2+2b^2)y+2ab^2}$ when $y=a$.

CHAPTER II.

RATIO.

18. Ratio is the relation which one Quantity bears to another with regard to magnitude. The former is called the **Antecedent,** the latter the **Consequent**; together they are called **Terms.**

19. The ratio of one quantity to another is expressed by writing the symbols side by side with a colon between, thus, $a : b$ (read a to b), or frequently in fractional form, thus, $\frac{a}{b}$.

20. Ratio may be treated either by considering the magnitudes themselves or the numbers which represent them. The former is the geometrical method followed by Euclid, Book V., which is the most logical treatise on ratio extant. The definition in Art. 18 is taken from that work as being the most complete and accurate which can be given. But Algebra deals with magnitudes only by means of the numbers which represent them, and Euclid's definition is not appropriate for that purpose; we are therefore compelled to give a different definition, which, though logically inferior to Euclid's, is still such as will enable us to treat the subject numerically.

21. The ratio of one quantity to another is the number or fraction which represents the former when the latter is taken as the unit.

Thus the ratio of 6 to 3 is 2, for 6 contains 3 twice; therefore 6 is represented by *two* when 3 is represented by *one*. Similarly the ratio of 3 feet to $2\frac{1}{2}$ inches is $14\frac{2}{5}$, etc. It should be carefully observed that ratio exists only between *quantities of the*

same kind, for the unit of measurement must evidently be of the same nature as the quantity to be measured.

22. Since the values of ratios are measured by fractions we are enabled at once to add, subtract, multiply and divide ratios by the rules which govern these operations in fractions, and all theorems which have been proved for fractions are equally true for ratios. For example,

The terms of a ratio may be multiplied or divided by the same number without changing its value.

In this connection see Arts. 151 and 169–171 of Part I.

23. A ratio is said to be "a ratio of greater inequality," "a ratio of equality," or "a ratio of less inequality," according as the antecedent is greater than, equal to, or less than the consequent.

In connection with this definition only the numerical values of the antecedent and consequent are to be considered, otherwise it would be inconsistent with previous definitions. For example, $3:4$ is a ratio of less inequality, since 3 is less than 4; but if this restriction were removed, $-3:-4$ would be a ratio of greater inequality, since -3 is algebraically greater than -4. But the value of the former ratio is $\frac{3}{4}$ and that of the latter $\frac{-3}{-4}=\frac{3}{4}$, *i.e.*, a ratio of greater inequality would be equal to a ratio of less inequality, which is absurd.

In the Theorems which follow the terms of the various ratios are considered positive.

24. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same positive quantity to both its terms.*

Let $a:b$ be the original ratio, and let $a+x:b+x$ be the ratio formed by adding the same positive quantity to both its terms.

Then $$\frac{a}{b}-\frac{a+x}{b+x}=\frac{x(a-b)}{b(b+x)},$$

and this result is positive or negative as a is greater or less than b.

Therefore, if $a > b$, $$\frac{a}{b} > \frac{a+x}{b+x},$$

and if $a < b$, $$\frac{a}{b} < \frac{a+x}{b+x},$$

which proves the proposition.

25. *A ratio of greater inequality is increased, and a ratio of less inequality is diminished, by subtracting the same positive quantity from both its terms.*

The proof is similar to that given in the last Art.

26. Ratios are compounded by multiplying together the fractions which represent them.

Thus the ratios $a : b$ and $c : d$ when compounded give $ac : bd$ as the resulting ratio.

27. The **Duplicate Ratio** of two quantities is the square of their ratio, and the **Triplicate Ratio** is the cube of their ratio.

Thus the duplicate ratio of $a : b$ is $a^2 : b^2$, and the triplicate ratio $a^3 : b^3$.

28. If there be three quantities such that the ratio of the first to the second equals the ratio of the second to the third, then the ratio of the first to the third is the duplicate ratio of the first to the second.

Let a, b, c be the three quantities,

then $$\frac{a}{b} = \frac{b}{c}, \text{ and } \therefore \left(\frac{a}{b}\right)^2 = \frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c},$$

which proves the proposition.

29. If there be four quantities such that the ratio of the first to the second, the second to the third, and the third to the fourth, are all equal, then the ratio of the first to the fourth is the triplicate ratio of the first to the second.

The proof is similar to that given in the last Art.

30. The **Subduplicate** and **Subtriplicate Ratios** of two

quantities are the square and cube roots respectively of their ratio; but these terms are now seldom used.

31. The **Inverse Ratio** of two quantities is the ratio of the second to the first. It may easily be shown to be the same as the ratio of their reciprocals; hence inverse ratio is often called **Reciprocal Ratio.**

32. *A ratio is increased by compounding it with a ratio of greater inequality, and diminished by compounding it with a ratio of less inequality.*

Let $a:b$ be compounded with $x:y$, then the resulting ratio $ax:by >$ or $< a:b$ according as $x >$ or $< y$.

For $$\frac{ax}{by}-\frac{a}{b}=\frac{a(x-y)}{by};$$

and this result is positive or negative according as $x >$ or $< y$, which proves the proposition.

FUNCTIONAL NOTATION.

33. The symbol $f(x)$ has already been used to denote a function of x. In the same way $f(x, y)$ may be used to denote a function of x and y, $f(x, y, z)$ to denote a function of the three quantities x, y and z, etc. The **form** of a function is the particular manner in which the quantities are involved. Different functions of the same quantities are denoted by using different letters before the brackets enclosing the quantities; thus $F(x, y)$ and $f(x, y)$ denote different functions of the same quantities, x and y. Sometimes a subscript or other distinguishing mark is used, thus, $F_1(x, y)$, $F_2(x, y)$, etc.

Again, if the form of $F(x, y)$ be given, $F(m, n)$ may be written by changing x and y in the given expression into m and n respectively; thus, if $F(x, y)$ denotes $ax^2+2hxy+by^2$, then $F(m, n)$ denotes $am^2+2hmn+bn^2$.

34. *If $F(x, y)$ denotes a homogeneous function of x and y of r dimensions, and if for x and y in this function we substitute mz and nz, the result will be $z^r . F(m, n)$.*

Let $F(x, y) = ax^r + bx^{r-1}y + cx^{r-2}y^2 + dx^{r-3}y^3 + \text{etc.},$
$$= am^rz^r + bm^{r-1}nz^r + cm^{r-2}n^2z^r + \text{etc.},$$
$$= z^r(am^r + bm^{r-1}n + cm^{r-2}n^2 + \text{etc.})$$
$$= z^r . F(m, n).$$

Cor. 1.—The above may easily be extended to any number of quantities; for let $F(x, y, z, \ldots)$ be a homogeneous function of $x, y, z, \ldots$ of r dimensions, and let $kx^uy^vz^w\ldots$ be any term in this function, then for $x, y, z, \ldots$ substituting $mt, nt, pt, \ldots$ we get $k(mt)^u(nt)^v(pt)^w\ldots$ or $km^un^vp^w\ldots t^{u+v+w\ldots}$, which proves the proposition.

Cor. 2.—If $x = my$, then $F(x, y)$ becomes $y^r . F(m, 1)$.

35. The two following theorems are extensions of Arts. 169 and 170, Part I. The student should carefully study what is there given under the heading "Theorems in Fractions," with the exercise following it, before reading what follows. He should then exercise himself by writing down numerous equalities of the kind included in the following Arts., testing them in each case by independent work.

36. *If there be two equal ratios, the ratio of any two homogeneous functions of the same number of dimensions of the terms of the first ratio is equal to the ratio of the same functions of the terms of the second ratio.*

Let $\dfrac{a}{b} = \dfrac{c}{d}$ be the equal ratios; $F(a, b)$, $f(a, b)$ the homogeneous functions, each of r dimensions; put each ratio $= m$, then $a = mb$ and $c = md$.

Then $$\frac{F(a, b)}{f(a, b)} = \frac{F(mb, b)}{f(mb, b)} = \frac{b^rF(m, 1)}{b^rf(m, 1)} = \frac{F(m, 1)}{f(m, 1)} \quad \begin{cases}\text{Art. 34,} \\ \text{Cor. 2.}\end{cases}$$

and $$\frac{F(c, d)}{f(c, d)} = \frac{F(md, d)}{f(md, d)} = \frac{d^rF(m, 1)}{d^rf(m, 1)} = \frac{F(m, 1)}{f(m, 1)}.$$

Therefore $$\frac{F(a, b)}{f(a, b)} = \frac{F(c, d)}{f(c, d)}.$$

37. *If there be any number of equal ratios, the ratio of any homogeneous function of r dimensions of the antecedents to the same function of the consequents is equal to the r^{th} power of one of the equal ratios.*

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots.$ be the equal ratios.

Let $F(a, c, e, \ldots.)$ be any homogeneous function of r dimensions of the antecedents; $F(b, d, f, \ldots.)$ the same function of the consequents; put each of the equal ratios $= m$, then $a = mb$, $c = md$, etc.

Then $$\frac{F(a, c, e, \ldots.)}{F(b, d, f, \ldots.)} = \frac{m^r . F(b, d, f, \ldots.)}{F(b, d, f, \ldots.)} = m^r, \qquad \begin{cases} \text{Art. 34,} \\ \text{Cor. 1.} \end{cases}$$

which proves the proposition.

EXERCISE II.

1. Write down the duplicate ratio of $5 : 7$ and the subduplicate ratio of $289 : 400$.

2. Which is the greater of the ratios, $9 : 10$ or $10 : 11$? $x : x + 1$ or $x + 1 : x + 2$? $a^3 + b^3 : a^2 + b^2$ or $a^2 + b^2 : a + b$?

3. Compound the ratios $8 : 11$ and $33 : 40$; $a : b$, $b : c$ and $c : a$.

4. Two numbers are in the ratio $5 : 7$, and if 33 be added to each the resulting numbers are in the ratio $8 : 11$. Find the numbers.

5. Find the ratio of x to y from each of the following equations:

(1) $ax - by = cx + dy$. (2) $3x^2 - 7xy = 6y^2$.

(3) $2x^2 - 5xy + 2y^2 = 0$. (4) $mx + ny = a(mx - ny)$.

6. The sum of two numbers is 100, and their ratio $7 : 13$. Find the numbers.

7. If 5 men and 6 boys do as much work as 7 men and 2 boys, and 40 men and 15 boys together earn $114 per day, find the wages of a man per day.

8. The difference between the number of chapters in the Old and in the New Testaments is 669, and if one were added to the number in each their ratio would be that of $310 : 87$. How many chapters are there in each?

9. What must be added to each term of the ratio $a : b$ to make it equal to the ratio $c : d$?

10. If the ratios $a : b$ and $b : c$ are equal, then $a : c$ is the duplicate ratio of $a : b$.

11. If the ratios $a : c$ and $c : b$ are equal, then $a : b$ is the duplicate ratio of $a + c : b + c$.

12. Prove that a ratio is increased or diminished by adding to its terms the corresponding terms of a greater or a lesser ratio.

13. If the ratios $a : b$, $c : d$, $e : f$ be in order of magnitude, then $a + c + e : b + d + f$ is less than $a : b$, but greater than $e : f$. Extend this principle to any number of ratios.

14. If $a : b = b : c$, then

$$a : a + b = a - b : a - c \text{ and } (a^2 + b^2)(b^2 + c^2) = (ab + bc)^2.$$

15. Which is the greater ratio,

$$a^2 - ab + b^2 : a^2 + ab + b^2 \text{ or } a^4 - a^2b^2 + b^4 : a^4 + a^2b^2 + b^4,$$

a and b having like signs?

16. If $a : b + c = m : n$ and $b : c + a = p : q$, find the ratio $c : a + b$.

17. The rates of two trains, A and B, are as $m : n$, and the lengths of their journeys are as $p : q$. It takes train B t hours longer to make its journey than it does train A. Find the time of each.

18. The rates of two trains are as $m : n$. They start at the same time from opposite ends of the same road; show that the ratio of the times they take after meeting to finish the journey is $n^2 : m^2$.

19. The arms of a pair of scales are of unequal length; a parcel placed in each pan in succession balances 49 and 64 lbs. respectively. Find its true weight and the ratio of the arms of the balance.

20. Each of two vessels contains a mixture of wine and water. A mixture formed of two measures from the first and one from the second contains wine and water in the ratio $56:79$, but if one measure be taken from the first and two from the second the ratio is $58:77$. Find the ratio of wine to water in each vessel.

21. If m gold coins are equal in weight to n silver coins, and p of the former equal in value q of the latter, compare the values of equal weights of gold and silver.

22. If m gold coins placed side by side reach as far as n silver ones, and p of the former are together as thick as q of the latter, and the values of equal bulks of gold and silver are as $r:s$, compare the values of a gold and a silver coin.

23. A street railway runs along an incline, and the ratio of the rates of a car up and down is $2:3$. The cars leave each terminus every ten minutes. At what intervals of time will a car going up meet the successive cars coming down, and *vice versa?*

24. A straight line is divided into two parts in the ratio $p:q$, and again in the ratio $r:s$. The distances between the points of section is a. Find the length of the line.

25. A straight line is divided into three parts in the ratio $p:q:r$. Find the ratio of the segments into which the middle point of the line divides the middle part.

38. A certain class of equations which occur very frequently in the higher mathematics may conveniently be discussed in connection with the subject of this chapter. Suppose we have given the single equation $ax+by=0$, the values of x and y are evidently indeterminate, since any value whatever may be assigned to one letter, and then a corresponding value may be obtained for the other. If, however, the different solutions be examined it will be found that the ratio of the values of x and y is constant, whatever value may be assigned to one of the letters. The equation may be written $a\left(\frac{x}{y}\right)+b=0$, from which $\frac{x}{y}=-\frac{b}{a}$. In fact, the original equation is not properly an equation between *two* un-

knowns, x and y, but an equation with *one* unknown, viz., the ratio $x:y$. This fact will be more clearly perceived if we take the two equations, $ax+by=0$ and $a'x+b'y=0$, when it will be found that the second equation will not assist in determining exact values for x and y, but will be inconsistent with the former unless $\frac{a'}{a}=\frac{b'}{b}$. If this condition be fulfilled, the second equation is a mere repetition of the first; if not, $x=0$ and $y=0$ is the only solution.

39. Between three quantities, x, y and z, two independent ratios exist, viz., $x:z$ and $y:z$. A third ratio, $x:y$, might be written, but its value is dependent on the other two. Both ratios may conveniently be expressed thus, $x:y:z$, which form has the additional advantage of representing the ratios of *any two* of the three quantities. An equation of the form $ax+by+cz=0$ may be considered an equation between two unknowns, viz., the ratios $x:z$ and $y:z$, as will immediately appear from dividing through by z. If, then, two such equations be given, the values of the two ratios may be determined.

Ex. 1.—Given $\left.\begin{aligned} ax+by+cz=0,\\ a'x+b'y+c'z=0,\end{aligned}\right\}$ to find the ratios $x:y:z$.

Dividing each of the equations through by z, and solving for $\frac{x}{z}$ and $\frac{y}{z}$, we get

$$\frac{x}{z}=\frac{bc'-b'c}{ab'-a'b}$$

and
$$\frac{y}{z}=\frac{ca'-c'a}{ab'-a'b}.$$

These results may be written in the more symmetrical form,

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b},$$

which gives the value of the ratios required.

Ex. 2.—To find the condition that the equations,

$$\left.\begin{aligned} ax+\ by+\ cz&=0,\\ a'x+b'y+c'z&=0,\\ a''x+b''y+c''z&=0,\end{aligned}\right\}$$

may be satisfied by the same values of x, y and z.

Writing the ratios $x:y:z$ from the second and third equations we get

$$\frac{x}{b'c''-b''c'}=\frac{y}{c'a''-c''a'}=\frac{z}{a'b''-a''b'}.$$

Now divide the terms of the first equation by these fractions in succession, which is merely dividing through by the same quantity, though in different forms. The result is

$$a(b'c''-b''c')+b(c'a''-c''a')+c(a'b''-a''b')=0.$$

If this condition be fulfilled the equations are satisfied by

$$x=k(b'c''-b''c'),\quad y=k(c'a''-c''a'),\quad z=k(a'b''-a''b'),$$

where k is any multiplier, since it is evident that the ratios of these values are the same as the ratios $x:y:z$ originally given. If the above condition be not fulfilled, then the only values which will satisfy the equations are $x=y=z=0$, which evidently satisfy any similar set of equations.

40. The examples of the preceding Art. are of great importance, and the student should be able to write the ratios from any similar set of equations without going through the successive steps of the solution. The results may easily be remembered as follows:

Write the equations one above the other, then omitting the coefficients of each letter in turn the coefficients of the other two form a square as below:

$$\begin{matrix} b,\ c, & \quad c,\ a, & \quad a,\ b,\\ b',\ c', & \quad c',\ a', & \quad a',\ b',\end{matrix}$$

the letters a, b, c, a', b', c', following each other in the usual cir-

cular order. The letters of these squares form the denominators of x, y and z respectively by taking the difference of the products of their diagonals, *beginning with the diagonal drawn downwards to the right.* The signs of the coefficients must be taken in connection with the coefficients themselves.

Ex. 1.—Write the ratios $x : y : z$ from the equations,

$$2x - 3y + z = 0,$$
$$x + 2y - z = 0.$$

Result:

$$\frac{x}{(-3)(-1)-(2)(1)} = \frac{y}{(1)(1)-(2)(-1)} = \frac{z}{(2)(2)-(-3)(1)},$$

or
$$\frac{x}{1} = \frac{y}{3} = \frac{z}{7}.$$

Ex. 2.—Solve equations:

$$ax + by + cz = 0,$$
$$x + y + z = 0,$$
$$a^2x + b^2y + c^2z + (a-b)(b-c)(c-a) = 0.$$

Writing the ratios from first and second equations,

$$\frac{x}{b-c} = \frac{y}{c-a} = \frac{z}{a-b}.$$

Dividing the terms of the third equation by these fractions,

$$a^2(b-c) + b^2(c-a) + c^2(a-b) + (a-b)(b-c)(c-a) \, . \, \frac{b-c}{x} = 0.$$

Dividing through by $-(a-b)(b-c)(c-a)$,

$$1 - \frac{b-c}{x} = 0 \quad \text{or} \quad x = b - c,$$

from which the values of y and z may be written, since the ratios $x : y : z$ are known.

Sometimes it is convenient to combine other quantities with x, y and z, and then to write the ratios of the resulting expressions as in the following example:

Ex. 3.—Solve equations:

$$\left.\begin{aligned} x+y+z&=a+b+c,\\ ax+by+cz&=ab+bc+ca,\\ (b-c)x+(c-a)y+(a-b)z&=0. \end{aligned}\right\}$$

The equations may be written,

$$(x-b)+\ (y-c)+\ (z-a)=0,$$
$$a(x-b)+b(y-c)+c(z-a)=0,$$
$$(b-c)(x-b)+(c-a)(y-c)+(a-b)(z-a)=bc+ca+ab-a^2-b^2-c^2.$$

From first and second equations,

$$\frac{x-b}{c-b}=\frac{y-c}{a-c}=\frac{z-a}{b-a},$$

and then from third equation,

$$(b-c)^2+(c-a)^2+(a-b)^2=(bc+ca+ab-a^2-b^2-c^2)\frac{(b-c)}{x-b}.$$

Therefore $$2=-\frac{b-c}{x-b}$$

or $$x=\frac{b+c}{2},$$

from which the values of y and z may be written from symmetry.

41. Commensurable Quantities are those which have a common measure, or those which are capable of being expressed in terms of the same unit. **Incommensurable Quantities** are those which have no common measure, or are not capable of being expressed in terms of the same unit.

A good example of incommensurable quantities is furnished by the side of a square and its diagonal. There is no unit of length that is contained an exact number of times in each. If the side be divided into 10 equal parts, the diagonal will contain more than 14 such parts, but less than 15; if it be divided into 100 equal parts, the diagonal will contain more than 141 but less than 142 such parts, and so on to any extent. Similarly, if the

diagonal be divided into an equal number of exact parts, the side will *not* contain an exact number of such parts. All this is briefly expressed by saying that the quantities are incommensurable.

42. The relative greatness of two magnitudes is in no way dependent on the manner in which they may be represented by symbols. The diagonal of a square admits of being compared, in regard to its length, with the side, even though they can not be represented numerically in terms of the same unit. The definition of ratio, therefore, of Art. 21 is not strictly appropriate for incommensurable quantities, and it is in this particular that it is inferior to that of Euclid. But though the ratio of two incommensurable quantities can not be *exactly* expressed by numbers, it can be expressed to any required degree of accuracy, as is shown in the following Art. The ratio between two incommensurable quantities is called an **Incommensurable Ratio.**

43. *If two quantities are incommensurable, a fraction may be found which will represent their ratio to any required degree of accuracy.*

For let a and b represent the two quantities; let b be divided into n equal parts, and let x represent one of those parts; then $b=nx$. Also let a be greater than mx, but less than $(m+1)x$; then $\frac{a}{b}>\frac{m}{n}$, but $<\frac{m+1}{n}$; then the difference between $\frac{a}{b}$ and $\frac{m}{n}$ is less than $\frac{1}{n}$. Therefore by taking n large enough the fraction $\frac{m}{n}$ differs from the exact ratio of a to b by less than any assignable quantity.

In such examples $\frac{m}{n}$ and $\frac{m+1}{n}$ are said to be the *limits* between which the true value of the ratio lies.

44. *Two incommensurable ratios are equal providing they always lie between the same limits, however small the difference between those limits may be.*

For let $a:b$ and $c:d$ be the two ratios whose values each lie between $\frac{m}{n}$ and $\frac{m+1}{n}$; then the difference between those ratios is less than $\frac{1}{n}$, and by taking n large enough this difference may be made less than any assigned difference between the ratios; and since there can be no assigned difference between the ratios, they must be equal.

EXERCISE III.

1. Given $\left.\begin{aligned} x+y+z&=0, \\ 2x+3y+4z&=0, \end{aligned}\right\}$ find the ratios $x:y:z$.

2. Given $\left.\begin{aligned} x&=ay+bz, \\ y&=bz+cx, \end{aligned}\right\}$ find the ratios $x:y:z$.

3. Solve equations:

$$2x+y-z=0,$$
$$x-2y-3z=0,$$
$$x^2+xy+y^2=84.$$

4. Solve equations:

$$2x-3y+4z=0,$$
$$3x-y-2z=0,$$
$$x^3+y^3+z^3=5439.$$

5. If $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$, and $(x+a)l+(y+b)m+(z+c)n=p$ then each fraction $=\frac{x^2+y^2+z^2-a^2-b^2-c^2}{p}$.

6. If
$$\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}=0,$$
and
$$\frac{x}{a(b-c)}+\frac{y}{b(c-a)}+\frac{z}{c(a-b)}=0,$$
then
$$\frac{a(b-c)^2}{x}=\frac{b(c-a)^2}{y}=\frac{c(a-b)^2}{z}.$$

7. If $\dfrac{x^2 - yz}{x(1 - yz)} = \dfrac{y^2 - zx}{y(1 - zx)}$, and x, y be unequal, then each fraction $= \dfrac{z^2 - xy}{z(1 - xy)} = x + y + z = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$.

8. If
$$cx + ay + bz = 0,$$
$$bx + cy + az = 0,$$
then
$$\frac{1}{a}(x^2 - yz) = \frac{1}{b}(y^2 - zx) = \frac{1}{c}(z^2 - xy)$$
and
$$(b - c)(x^2 - yz) + (c - a)(y^2 - zx) + (a - b)(z^2 - xy) = 0.$$

9. If
$$\frac{x}{b+c} + \frac{y}{c+a} + \frac{z}{a+b} = 0,$$
and
$$\frac{x}{b-c} + \frac{y}{c-a} + \frac{z}{a-b} = 0,$$
then
$$\frac{x}{(a^2-bc)(b^2-c^2)} = \frac{y}{(b^2-ca)(c^2-a^2)} = \frac{z}{(c^2-ab)(a^2-b^2)}$$
and
$$\frac{x}{a^2-bc} + \frac{y}{b^2-ca} + \frac{z}{c^2-ab} = 0.$$

Solve the following equations:

10.
$$x + y + z = 0,$$
$$ax + by + cz = (c - b)x + (a - c)y + (b - a)z$$
$$= a^2 + b^2 + c^2 - ab - bc - ca.$$

11.
$$x + y + z = 3(a + b + c),$$
$$ax + by + cz = 3(ab + bc + ca),$$
$$(b - c)x + (c - a)y + (a - b)z = bc + ca + ab - a^2 - b^2 - c^2.$$

12.
$$bx + cy + az = cx + ay + bz$$
$$= a^2 + b^2 + c^2,$$
$$x + y + z = a + b + c.$$

13.
$$x + y + z = a + b + c,$$
$$ax + by + cz = ab + bc + ca,$$
$$(b + c)x + (c + a)y + (a + b)z = a^2 + b^2 + c^2 + ab + bc + ca.$$

14. $$\frac{ax+by}{a-b}=\frac{by+cz}{b-c}=\frac{cz+ax}{c-a},$$

$$a^3x+b^3y+c^3z+(a-b)(b-c)(c-a)=0.$$

15. $$\frac{2x+3y-4z}{x+5}=\frac{3x+4y-2z}{5x}=\frac{4x+2y-3z}{4x-1}=\frac{x+y-z}{6}.$$

16. $$\frac{a^3x}{y^2z^2}=\frac{b^3y}{z^2x^2}=\frac{c^3z}{x^2y^2}=1.$$

17. If $\sqrt{al}\pm\sqrt{bm}\pm\sqrt{cn}=0$, the two values of $y:z$ obtained from

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0,$$

$$lx+my+nz=0,$$

will be equal.

18. Find the condition that the equations,

$$ax+hy+gz=0,\ hx+by+fz=0,\ gx+fy+cz=0,$$

may be satisfied by the same values of x, y and z.

19. Find the condition that the equations,

$$ax+cy+bz=0,\ bx+ay+cz=0,\ cx+by+az=0,$$

may be satisfied by the same values of x, y and z.

20. If the equations,

$$ax+by+cz=0,\ a'x+b'y+c'z=0,\ a''x+b''y+c''z=0,$$

are satisfied by the same values of x, y and z, then

$$ax+a'y+a''z=0,\ bx+b'y+b''z=0,\ cx+c'y+c''z=0,$$

are also satisfied by the same values of x, y and z.

21. If $$\frac{x}{y+z}=a,\ \frac{y}{z+x}=b,\ \frac{z}{x+y}=c,$$

find the relation between a, b and c, and show that

$$\frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ca)}=\frac{z^2}{c(1-ab)}.$$

22. If $x=cy+bz$, $y=az+cx$, $z=bx+ay$,

prove $$\frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}=\frac{z^2}{1-c^2}.$$

23. If
$$\frac{(b^2c^2+1)x+b^2+c^2}{bc}=\frac{(c^2a^2+1)x+c^2+a^2}{ca}=\frac{(a^2b^2+1)x+a^2+b^2}{ab},$$
and a, b and c are all unequal, then each fraction is equal to $(bc+ca+ab)x-1$ and to $\dfrac{(b+c)(c+a)(a+b)x}{a+b+c}$, and $x=\dfrac{1}{ab}+\dfrac{1}{bc}+\dfrac{1}{ca}$.

24. If $$\frac{bx+ay-cz}{a^2+b^2}=\frac{cy+bz-ax}{b^2+c^2}=\frac{az+cx-by}{c^2+a^2},$$

then $$\frac{x+y+z}{a+b+c}=\frac{ax+by+cz}{ab+bc+ca}.$$

25. If $$\frac{x}{a(y+z)}=\frac{y}{b(z+x)}=\frac{z}{c(x+y)},$$

then $$\frac{x^2-y^2}{a-b}=\frac{y^2-z^2}{b-c}=\frac{z^2-x^2}{c-a}$$

and $$\frac{xy}{ab}(a-b)+\frac{yz}{bc}(b-c)+\frac{zx}{ca}(c-a)=0.$$

26. If $$\frac{x}{l(mb+nc-la)}=\frac{y}{m(nc+la-mb)}=\frac{z}{n(la+mb-nc)},$$

then $$\frac{ny+mz}{a}=\frac{lz+nx}{b}=\frac{mx+ly}{c}$$

and $$\frac{x}{l}(mb-nc)+\frac{y}{m}(nc-la)+\frac{z}{n}(la-mb)=0.$$

27. If $aX+bY+cZ=0$ and $a_1X+b_1Y+c_1Z=0$ where
$$X=ax+a_1x_1+a_2,\quad Y=bx+b_1x_1+b_2,\quad Z=cx+c_1x_1+c_2,$$
then $$X^2+Y^2+Z^2=\frac{\{a_2(bc_1-b_1c)+b_2(ca_1-c_1a)+c_2(ab_1-a_1b)\}^2}{(bc_1-b_1c)^2+(ca_1-c_1a)^2+(ab_1-a_1b)^2}.$$

CHAPTER III.

PROPORTION.

45. Proportion is the equality of two ratios. Four quantities are said to be in proportion when the ratio of the first to the second equals the ratio of the third to the fourth, and the quantities themselves are called proportionals. The first and fourth quantities are called **Extremes,** and the remaining two are called **Means.**

46. The equality of two ratios may be indicated in various ways. Thus, if $a : b$ and $c : d$ are the two ratios, $a : b :: c : d$ (read, as a is to b so is c to d), $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$, indicates that the ratios are equal and that the four quantities are in proportion. Similarly the equality of three or more ratios may be expressed, thus, $a : b :: c : d :: e : f$, or $a : b = c : d = e : f$, or the fractional form may be used as before; or again, the antecedents may all precede the sign of equality, written but once, thus, $a : c : e = b : d : f$, and so on to any extent.

47. Four quantities are required to form a proportion; but one may be repeated, thus requiring only three different quantities. The quantities forming a ratio must be of the same kind, but those forming the first ratio may be different from those forming the second; but if only three different terms are used, all must be of the same kind.

48. *If four quantities are proportionals, the product of the extremes is equal to the product of the means.*

Let a, b, c, d be the four quantities,

such that $$a : b :: c : d.$$

Then $$\frac{a}{b} = \frac{c}{d} \text{ by definition;}$$

therefore $$ad = bc.$$

This proposition enables us to change a proportion into an equation, and thus to find any one of four proportionals when the other three are given.

49. *If the product of two quantities equals the product of two others, the four quantities are proportionals, the factors of either product being taken for the extremes, and the factors of the other for the means.*

For if $$ad = bc,$$

then $$\frac{a}{b} = \frac{c}{d} \quad \text{and} \quad \frac{a}{c} = \frac{b}{d}.$$

Therefore $$a : b = c : d \text{ and } a : c = b : d.$$

Similarly $$b : a = d : c \text{ and } b : d = a : c.$$

50. From the four terms which form a proportion many other proportions may be formed by permutating and combining the terms in various ways. The most important changes are given in the following Art., together with the technical terms by which some of them are known.

51. If four quantities, a, b, c, d, are proportionals,

so that $$a : b :: c : d,$$

(1) They are proportional when taken inversely;

that is, $$b : a :: d : c. \qquad \textit{(Invertendo.)}$$

(2) They are proportional when taken alternately;

that is, $$a : c :: b : d. \qquad \textit{(Alternando.)}$$

(3) The first, together with the second, is to the second as the third, together with the fourth, is to the fourth;

that is, $a+b:b::c+d:d$. (*Componendo.*)

(4) The excess of the first above the second is to the second as the excess of the third above the fourth is to the fourth;

that is, $a-b:b::c-d:d$. (*Dividendo.*)

(5) The first is to its excess above the second as the third is to its excess above the fourth;

that is, $a:a-b::c:c-d$. (*Convertendo.*)

(6) The sum of the first and second is to their difference as the sum of the third and fourth is to their difference;

that is, $a+b:a-b::c+d:c-d$.

(7) Any equimultiples of the first and second are proportional to any equimultiples of the third and fourth;

that is, $ma:mb::nc:nd$.

Also, if equimultiples of the first and third, and of the second and fourth, be taken, they will be proportional;

that is, $ma:nb::mc:nd$,

where m and n are any real quantities.

(8) Like powers or roots of the four quantities are proportional;

that is, $a^m:b^m::c^m:d^m$,

where m is any real quantity.

The proof of these various propositions follows at once from the equality of the fractions which express the equal ratios.

Since $a:b::c:d$,

therefore $\frac{a}{b}=\frac{c}{d}$ by definition.

Divide unity by each of these equal quantities,

then $$\frac{b}{a}=\frac{d}{c}. \tag{1}$$

Therefore $b:a::d:c$.

Subtract each side of (1) from a unit,

then $$\frac{a-b}{a}=\frac{c-d}{c}.$$

Therefore $$\frac{a}{a-b}=\frac{c}{c-d},$$

or $$a : a-b :: c : c-d.$$

Similarly all the required results may easily be obtained.

Additional results may be obtained by a combination of the preceding principles; thus, applying the sixth principle to the second result of the seventh we get

$$ma+nb : ma-nb :: mc+nd : mc-nd.$$

52. If four quantities are proportional, then any two homogeneous functions of the first and second are proportional to the same functions of the third and fourth—all the functions being of the same number of dimensions.

This Theorem may be concisely expressed in symbols thus.

If $$a : b :: c : d,$$

then $$F(a, b) : f(a, b) :: F(c, d) : f(c, d),$$

the functions being homogeneous and of the same number of dimensions.

The proof is at once evident from Art. 36. See also a similar Art. on Variation (Art. 72). This proposition evidently includes all the results of Art. 51.

53. If $a : b :: b : c$, then a, b, c are said to be in continued proportion, and b is a **mean proportional** between a and c; also, c is said to be a third proportional to a and b. If a, b, c denote the lengths of straight lines, then, since in this case $ac=b^2$, b is the length of the side of a square which is equal in area to the rectangle contained by the lines a and c. Hence the last proposition of the Second Book of Euclid is equivalent to finding a mean proportional between two given quantities.

54. If $a : b :: b : c :: c : d$, then a, b, c, d are said to be in continued proportion, and b and c are two mean proportionals between a and d. In this case we have

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}.$$

Therefore $$\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{d} = \left(\frac{a}{b}\right)^3, \text{ or } \frac{a}{d} = \frac{a^3}{b^3},$$

from which $$b^3 = a^2 d.$$

Therefore b is the length of the edge of a cube equal in volume to a rectangular solid whose length and breadth are each a, and height d. To find this length by elementary geometry was one of the three famous problems of antiquity which could never be solved.

55. If $a : b :: c : d$ and $e : f :: g : h$,

then $$ae : bf :: cg : dh.$$

For $$\frac{a}{b} = \frac{c}{d} \text{ and } \frac{e}{f} = \frac{g}{h}.$$

Therefore $$\frac{ae}{bf} = \frac{cg}{dh},$$

or $$ae : bf :: cg : dh.$$

Cor.—If $a : b :: x : y$ and $b : c :: y : z$,

then $$a : c :: x : z.$$

This principle may evidently be extended to any number of quantities which are proportional to as many others. It is quoted by the words *ex æquali* or *ex æquo*.

56. It will be instructive to compare the test for proportion, or the equality of two ratios, laid down by Euclid, Bk. V., Def. 5, with that already given. Euclid's definition may be stated thus:

Four quantities are proportionals when, if any equimultiples whatever be taken of the first and third, and also any equimulti-

ples whatever of the second and fourth, the multiple of the third is always greater than, equal to, or less than, the multiple of the fourth, according as the multiple of the first is greater than, equal to, or less than, the multiple of the second.

This definition, which is somewhat unwieldy when expressed in words, will be more readily intelligible when expressed by symbols as follows:

If there be four magnitudes, A, B, C, D, such that

$$mC >, =, \text{ or } <, nD,$$

according as $$mA >, =, \text{ or } <, nB,$$

for all positive integral values of m and n, then A, B, C, D are said to be proportionals.

57. We shall now show that quantities which are proportional according to the algebraic test are proportional according to Euclid's test, and conversely.

1. Let a, b, c, d be proportionals algebraically.

Then $$\frac{a}{b} = \frac{c}{d} \text{ by definition.}$$

Multiplying each fraction by $\frac{m}{n}$

we get $$\frac{ma}{nb} = \frac{mc}{nd}.$$

Now, ma and mc are any equimultiples of the first and third, and nb and nd are any equimultiples of the second and fourth, and from the principles of fractions

$$mc >, =, \text{ or } <, nd,$$

according as $$ma >, =, \text{ or } <, nb,$$

which proves the proposition.

2. Let a, b, c, d be proportional according to the geometrical

definition, then shall they be proportional according to the algebraical definition.

For if $\frac{a}{b}$ be not equal to $\frac{c}{d}$, let $\frac{a}{b}$ be the greater; and let $\frac{n}{m}$ be less than $\frac{a}{b}$, but greater than $\frac{c}{d}$.

Then, since $\frac{a}{b} > \frac{n}{m}$, $\therefore\ ma > nb$;

and since $\frac{c}{d} < \frac{n}{m}$, $\therefore\ mc < nd$.

Now, of the four quantities, a, b, c, d, of the first and third equimultiples ma and mc have been taken, and of the second and fourth equimultiples nb and nd have been taken; and the multiple of the first is greater than the multiple of the second, but the multiple of the third is less than that of the fourth, which is contrary to the supposition that a, b, c, d are proportional according to the geometrical definition; therefore $\frac{a}{b}$ is not unequal to $\frac{c}{d}$, *i.e.*, they are equal, which proves the proposition.

EXERCISE IV.

1. Find a fourth proportional to 3, 5 and 15.

2. The second, third and fourth terms of a proportion are 12, 41 and $61\frac{1}{2}$; find the first.

3. Find a third proportional to $1+\sqrt{2}$ and $3+2\sqrt{2}$, and a mean proportional between $\sqrt{7}-\sqrt{5}$ and $11\sqrt{7}+13\sqrt{5}$.

4. What number must be added to each of the numbers 1, 3, 5 and 8 so that the results will be proportional?

5. Find a number which, added to 1 and to 11, will give results between which 12 is a mean proportional.

6. Given that $x+y : x-y :: a+b : a-b$, and m is a mean proportional between x and y, find x and y.

4

7. Three numbers are in continued proportion; the sum of the greatest and the least is 51, and the sum of the two greatest is 60. Find the numbers.

8. Given that the work done by $x-1$ men in $x+1$ days is to the work done by $x+3$ men in $x-2$ days as $20:21$, find x.

9. If four quantities are in continued proportion, the difference between the first and last is more than three times the difference between the other two.

10. If $(a^2+b^2)(b^2+c^2)=(ab+bc)^2$, then a, b, c are in continued proportion.

11. If a, b, c are in continued proportion,

then $$a+mb : a-mb :: b+mc : b-mc,$$

and $$\left(\frac{b}{c}+\frac{c}{a}\right) : \left(\frac{c}{a}+\frac{a}{b}\right) \text{ is a ratio of equality.}$$

12. What must be subtracted from each term of the ratio $a:b$ that the resulting ratio may be the duplicate of the original ratio?

13. Find two numbers whose sum, difference and product are proportional to s, d and p.

14. If $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{d}$, $\frac{d}{e}$ are proportionals,

then $$b^4+a^2c^2 : b^4-a^2c^2 :: d^4+c^2e^2 : d^4-c^2e^2;$$

and if a, b, d are also in continued proportion, then $d=e$.

15. If $b+c+d$, $c+d+a$, $d+a+b$, $a+b+c$ are proportionals,

then $$b^2+bc+c^2=a^2+ad+d^2.$$

16. What must be added to each of the four quantities a, b, c, d so that the results will be in proportion? Examine the case in which $a+d=b+c$.

17. If $a+b : m+n :: m-n : a-b$,

then $$a+m : b+n :: b-n : a-m;$$

and if a is greater than m, then b is greater than n.

18. If a, b, c, d are proportionals,

then $$a+d=b+c+\frac{(a-b)(a-c)}{a}.$$

19. If a, b, c, d are proportionals, and if a is the greatest, then d is the least; also, $a+d>b+c$ and $a^2+d^2>b^2+c^2$.

20. If the ratio of the difference of the antecedents of two ratios to the difference of the consequents is measured by the sum of the measures of the separate ratios, the antecedents are in the duplicate ratio of the consequents.

21. If x and y be such that, when added respectively to the antecedent and consequent of the ratio $a:b$, the resulting ratio is the reciprocal of that formed by adding them to the consequent and antecedent, then either $a=b$ or $x+y+a+b=0$.

22. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$,

then $$a:b::c:d.$$

23. If $$(pa+qb+rc+sd)(pa-qb-rc+sd)$$ $$=(pa-qb+rc-sd)(pa+qb-rc-sd),$$

then $$bc:ad::ps:qr,$$

and $$br:pd::as:qc;$$

and if either of the two sets, a, b, c, d, or p, q, r, s, are proportionals, the others are proportionals also.

24. Sold goods for \$24, losing as much per cent. as the goods cost; find the cost. What should be the cost for the selling price to be as great as possible?

25. The time which an express train takes to travel 180 miles is to that taken by an ordinary train as 9 : 14. The ordinary train loses as much time from stoppages as it would take to travel 30 miles without stopping. The express train loses only half as much time as the other by stopping, and travels 15 miles an hour faster. What are their rates respectively?

26. To 300 lbs. of a mixture containing 2 parts of zinc and 3 of copper and 4 of tin was added 200 lbs. of another mixture of

the same metals, when it was found that the proportions were now as 3, 4, 5. What were the proportions in the added mixture?

27. Each of two vessels contains 10 gals. of alcohol and water, the first in the ratio of $3 : 2$ and the second in the ratio $1 : 4$; how many gallons must be poured from the first into the second, and then the same amount from the second into the first, to leave 5 gallons of alcohol in the first vessel?

28. The first of two vessels is filled with wine and water in the ratio $m . n$; the second in the ratio $p : q$ Their contents being mixed, the resulting liquid contains wine and water in the ratio $r : s$; find the ratio of the volumes of the vessels.

29. If a, b, c are in continued proportion,

then
$$(a-b)^2 : b(a-c) :: a(b-c) : c(a+b);$$
$$a^2(a-b+c) : a^2+ab+b^2 :: a^2-ab+b^2 : a+b+c;$$
$$\frac{(b+c)^2}{b-c}+\frac{(c+a)^2}{c-a}+\frac{(a+b)^2}{a-b}=\frac{4b}{a-c}(a+b+c).$$

30. If a, b, c, d are in continued proportion,

then
$$(a-c)(b-d)-(a-d)(b-c)=(b-c)^2;$$
$$(b-c)^2+(c-a)^2+(d-b)^2=(a-d)^2;$$
$$(ad+bc)(a+b+c+d)(a-b-c+d)=2(ab-cd)(ac-bd).$$

31. Brass is an alloy of copper and zinc; bronze is an alloy containing 80 per cent. of copper, 4 of zinc and 16 of tin. A fused mass ot brass and bronze is found to contain 74 per cent. of copper, 16 of zinc and 10 of tin; find the percentages of copper and zinc in the composition of brass.

32. A and B are partners in a business in which their interests are in the ratio $a : b$. They admit C to the partnership, without altering the whole amount of capital, in such a way that the interests of the three partners are then equal. C pays \$$c$ for the privilege. How is this sum to be divided between A and B, and what capital had each in the business originally?

CHAPTER IV.

VARIATION.

58. The object of the present chapter is to present the principle of proportion in a slightly different form, and one which is specially adapted to complicated and intricate problems. The difficulty usually experienced by students in this method of treatment of the subject will be removed by giving careful attention to the meaning of the technical terms employed, as explained by a few simple examples.

59. The concrete magnitudes to which mathematics are applied are so related to each other that a change in the one frequently produces a corresponding change in another, or, in mathematical language, one is a function of the other. Thus the circumference, the surface and the volume of a sphere will each be changed if the length of the radius be changed; *i.e.*, they are each functions of the radius. The ratio of the circumference to the diameter, however, will not be changed by changing the radius; the value of this ratio is therefore a "constant," whilst the radius, circumference, surface and volume are "variables." If we conceive of different values being given in succession to the radius, it may be called the "independent" variable, whilst the other quantities, from their values being dependent upon the value of the radius, are called "dependent" variables.

60. The value of one quantity may depend upon the value of several others; thus the area of a triangle depends upon, or is a function of, the base and altitude. When one quantity increases another may diminish; thus the time necessary to travel a given

distance diminishes as the speed increases. The number of ways in which the value of one quantity may be connected with, or be dependent on, others is without limit. In any particular problem the exact nature of the dependence or connection must first be clearly conceived in the mind, and then accurately expressed in mathematical symbols.

61. If a single equation be given, containing two unknowns, x and y, we may give any value we please to one of them, and then corresponding values of the other may be determined. In such cases either quantity may be considered a function of the other, and the quantities themselves are variables. Consider the following simple examples:

$$y=2x,\quad y=4x^2,\quad y=ax+b,\quad y=ax^2+bx+c,\quad y=2^x.$$

In each case the value of y is known when that of x is known; we therefore say that x and y are variables, that y is a function of x, and that x is the independent variable, and y the dependent variable.

62. We have now given a number of examples of both concrete and symbolical quantities, of which the value of one varies (*i.e.*, changes) when the other varies or changes, and therefore, in popular language, one may be said to vary as the other. But the word "variation," in mathematical language, is restricted to one particular kind of change in value, or functional dependence between two quantities, which we shall now proceed to explain.

63. Variation is an abridged method of indicating proportion; its precise meaning is determined by the definition of the following Art.

64. One quantity is said to vary directly as another when the two are so connected that if one be increased or decreased in any ratio, the other is increased or decreased in the same ratio.

Thus, if the rate per cent. and the time be constant, the simple interest varies directly as the principal; for if P and p be two

sums of money, I and i the interest, $I : i = P : p$, the rate and time being constant.

Again, if $y = mx$ where m is a constant, y varies as x; for if y' and x' be simultaneous values of y and x, we have $y' = mx'$;

$$\therefore \frac{y}{y'} = \frac{mx}{mx'} = \frac{x}{x'}, \text{ or } y : y' = x : x'.$$

The symbol $\propto$ placed between two quantities signifies that the first varies as the second; thus $y \propto x$ is read y varies as x. The statement that one algebraic expression varies as another is often called an equation of variation.

65. One quantity is said to vary inversely as another when the two are so connected that if one be increased or diminished in any ratio, the other is diminished or increased in the same ratio.

Thus the time in which a given piece of work can be performed varies inversely as the number of men employed in it; for if the number of men be increased or decreased in any ratio, the time required will evidently be decreased or increased in the same ratio.

Again, if $y = \frac{m}{x}$ where m is any constant, $y \propto$ inversely as x; for if y' and x' be simultaneous values of y and x, we have $y' = \frac{m}{x'}$, and therefore $\frac{y}{y'} = \frac{x'}{x}$ or $y : y' = x' : x$.

66. One quantity is said to vary jointly as two others when it varies as their product. The area of a triangle varies jointly as its base and altitude, for the area is always one-half the product of the base and height; whence the truth of the statement is evident from preceding definitions. If A denote the area, b the base and h the height, we have $A = \frac{1}{2}bh$, from which it is evident that A increases or decreases in the same ratio as the product bh. We also see from this example that when the relation of quantities is expressed in symbols it is much easier to determine the nature of the change in one produced by a change in another, than from considering the magnitudes themselves.

67. One quantity is said to vary directly as a second, and inversely as a third, when the first varies as the quotient of the second by the third. Thus the time required to travel a distance varies directly as the distance and inversely as the velocity; for the time is always equal to the quotient of the distance by the velocity, so that if the quotient be changed in any ratio, the time is changed in the same ratio.

Again, if $y = m\left(\frac{x}{z}\right)$ where m is constant, then $y \propto \frac{x}{z}$; for if $\frac{x}{z}$ be changed in any ratio, y is evidently changed in the same ratio.

68. Theorem I.—*If $y \propto x$, then $y = mx$ where m is constant for all values of x and y.*

Let x be changed to x', and in consequence let y become y',

then $$\frac{y}{y'} = \frac{x}{x'} \text{ by definition, Art. 64.}$$

Therefore $$y = \left(\frac{y'}{x'}\right)x = mx \text{ if } \frac{y'}{x'} = m.$$

Now, x' and y' are fixed numbers; therefore for all values of x and y we have $y = mx$, where m is a constant quantity.

The converse of this Theorem has already been proved (Art. 64).

69. The preceding Theorem, being the fundamental one of this part of the subject, deserves the most careful attention. The point which usually confuses a beginner is the change in meaning of the symbols y and x which takes place in the course of proof. In the enunciation y and x are evidently variables. Upon beginning the proof they are supposed to have some definite value; afterwards y' and x' are supposed to remain constant, whilst y and x are again susceptible of change.

70. Theorem II.—*If $y \propto x$ when z is constant, and $y \propto z$ when x is constant, then $y \propto xz$ when both x and z are variable.*

The variation of y depends upon the variations of x and z. Let the changes in the latter quantities take place separately.

First let x be changed to x', and in consequence let y become y_1; next let z be changed to z', and in consequence let y_1 become y', so that we have the following sets of simultaneous values of the variables:

$$y,\quad x,\quad z.$$
$$y_1,\quad x',\quad z.$$
$$y',\quad x',\quad z'.$$

From the first change of values we have

$$\frac{y}{y_1} = \frac{x}{x'},$$

and from the second, $$\frac{y_1}{y'} = \frac{z}{z'}.$$

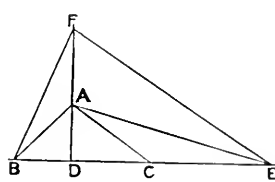
From these two equations, $$\frac{y}{y'} = \frac{xz}{x'z'};$$

that is, $$y \propto xz.$$

71. The following illustration will render the preceding Art. more easily intelligible.

The area of a triangle varies as the base when the height is constant, and the area varies as the height when the base is constant; and when both base and height vary, the area varies as their product.

Let ABC be any triangle. Denote its area by y, its base BC by x, and its height AD by z. First extend the base to any point E, and denote its length by x', and the area of the triangle ABE by y_1; then $\dfrac{y}{y_1} = \dfrac{x}{x'}$. Next increase the height DA to DF. Denote this height by z', and



the area of the triangle FBE by y'; then $\dfrac{y_1}{y'} = \dfrac{z}{z'}$. From these equations as before $\dfrac{y}{y'} = \dfrac{xz}{x'z'}$, which shows that the area varies as the product of the height and the base.

Again, the variations may be inverse, or one may be direct and the other inverse. A good example of the latter case is furnished by the change of pressure of a gas when the volume and temperature change. It is found by experiment that the pressure, p, of a gas varies as the "absolute temperature," t, when the volume, v, is constant, and inversely as the volume when the temperature is constant; that is,

$$p \propto t \quad \text{when } v \text{ is constant,}$$

and
$$p \propto \frac{1}{v} \quad \text{when } t \text{ is constant.}$$

From these equations $p \propto \frac{t}{v}$ when t and v are both variable, and by actual experiment this is found to be the case.

72. *If $y \propto x$, then any homogeneous function of x and y varies as any other homogeneous function of the same number of dimensions.*

Let $F(x, y)$, $f(x, y)$ denote any two homogeneous functions of x and y, each of r dimensions; and, since $y \propto x$, $\therefore\ y = mx$.

Then
$$\frac{F(x, y)}{f(x, y)} = \frac{F(x, mx)}{f(x, mx)} = \frac{x^r . F(1, m)}{x^r . f(1, m)} = \frac{F(1, m)}{f(1, m)} = \text{a constant.}$$

Therefore
$$F(x, y) \propto f(x, y).$$

73. If an equation of variation exists between two homogeneous functions of the same number of dimensions of x and y, then $y \propto x$.

Let $F(x, y)$, $f(x, y)$ denote two homogeneous functions, each of r dimensions, of which one varies as the other.

Let $y = mx$, then we have to show that the value of m does not change when x and y change.

Since
$$F(x, y) \propto f(x, y), \quad \therefore\ F(x, y) = k . f(x, y).$$

Therefore
$$F(x, mx) = k . f(x, mx);$$
$$\therefore\ x^r . F(1, m) = k . x^r f(1, m);$$
$$\therefore\ F(1, m) = k . f(1, m).$$

Hence the value of m is independent of the values of x and y, and is therefore a constant; $\therefore\ y \propto x$.

74. If $y \propto x \propto z$, then any homogeneous function of r dimensions of x and y varies as z^r.

Let $F(x, y)$ denote any homogeneous function of r dimensions, and, since x and y each $\propto z$,

$\therefore\ x = mz,\ y = nz$ where m and n are constants.

Then $$F(x, y) = F(mz, nz) = z^r \,.\, F(m, n) \propto z^r,$$ Art. 34

since $F(m, n)$ is constant.

Cor.—This principle may easily be extended to any number of variables, thus:

If $y \propto z \propto x \propto u \propto \dots$, then any homogeneous function of r dimensions of these variables varies as the r^{th} power of any one of them. For express all in terms of any one, thus $y = mu$, $z = nu$, etc.,

Then $$F(y \,.\, z \,.\, x \,.\, u \dots) = u^r \,.\, F(m, n \dots) \propto u^r.$$

Hence any homogeneous functions of the variables varies as any other homogeneous function of the same number of dimensions.

75. If two equations of variation exist between homogeneous functions of three variables, x, y, z, the functions belonging to the same equation being of the same number of dimensions, then $y \propto x \propto z$. Put $y = mz$ and $x = nz$, then we have to show that the values of m and n are independent of x, y and z. Proceeding as in Art. 73 we obtain two equations independent of x, y and z to determine the values of m and n.

REMARK.—It is not necessary to solve these equations, but only to observe that m and n must have fixed values independent of x, y and z.

76. The following examples illustrate the meaning of the preceding Arts. They also show the method which should be adopted for the solution of similar examples in the exercise which follows.

Ex. 1.—If $y \propto x$, then $2x^2 - 3xy \propto y^2 - 5x^2$.

Since $y \propto x$, $\therefore$ $y = mx$ where m is a constant.

Then $$\frac{2x^2 - 3xy}{y^2 - 5x^2} = \frac{2x^2 - 3mx^2}{m^2x^2 - 5x^2} = \frac{2 - 3m}{m^2 - 5} = \text{a constant.}$$

Therefore $$2x^2 - 3xy \propto y^2 - 5x^2.$$

Ex. 2.—If $x^3 - 3x^2y + y^3 \propto 4x^2y - 2xy^2$, then $y \propto x$.

Let $y = mx$, then we have to show that the value of m does not change when x and y change.

We have $x^3 - 3x^2y + y^3 = k(4x^2y - 2xy^2)$ where k is a constant.

Therefore $x^3(1 - 3m + m^3) = kx^3(4m - 2m^2)$.

$$\therefore \ 1 - 3m + m^3 = k(4m - 2m^2).$$

From this equation we see that the value of m is independent of the value of x and y; $\therefore$ $y \propto x$.

Ex. 3.—If $y \propto x \propto z$, then $x^3 - 2y^2z + 3xyz \propto z^3$.

Since both y and x vary as z, let $y = mz$, $x = nz$.

Then $$x^3 - 2y^2z + 3xyz = z^3(n^3 - 2m^2 + 3mn)$$
$$= z^3 \times \text{a constant.}$$

Therefore $$x^3 - 2y^2z + 3xyz \propto z^3.$$

Ex. 4.—If $y \propto x + z$ and $y^2 + zx \propto z^2 + xy$, then $y \propto x \propto z$.

Let $y = mz$, $x = nz$; we have then to show that m and n are constants for all values of x, y and z.

Since $$y \propto x + z, \quad \therefore \ y = k(x + z).$$

And since $$y^2 + zx \propto z^2 + xy, \quad \therefore \ y^2 + zx = l(z^2 + xy)$$

where k and l are constants.

Substituting the given values for y and x in these equations we get

$$mz = k(nz + z) \text{ or } m = k(n + 1);$$
$$m^2z^2 + nz^2 = l(z^2 + mnz^2) \text{ or } m^2 + n = l(1 + mn).$$

These two equations determine constant values for m and n, which proves the proposition.

REMARK.—It may be objected that in such examples m and n may have several different values, and are therefore not constants. The reply is that their values are restricted to certain definite quantities which do not change when x, y and z change; they are therefore constants within the meaning of that term as used in Variation.

77. The following examples show the application of the principles of variation to the solution of problems.

Ex. 1.—Given that $y \propto x$, and when $x=2$, $y=3$; find the value of y when $x=30$.

Since $y \propto x$, $\therefore$ $y=mx$ where m is constant for all values of x and y. Substituting the given values of x and y we get $3=2m$; $\therefore$ $m=\frac{3}{2}$; $\therefore$ $y=\frac{3}{2}x$, and when $x=30$, $y=\frac{3}{2}x=\frac{3}{2}\times 30=45$, the number required.

Ex. 2.—If 3 men working 5 days earn \$30, how much will 7 men earn in 9 days?

From the nature of the problem the amount earned varies jointly as the number of men and the time they work.

Let x, y, z denote the number of men, the time they work, and the amount earned respectively.

Then $$z \propto xy, \quad \therefore \; z=mxy.$$

Substituting 3, 5 and 30 for x, y and z we get

$$30=m\times 3\times 5.$$

From which $$m=2, \quad \therefore \; z=2xy.$$

Then substituting 7 and 9 for x and y in this last equation we get $z=2\times 7\times 9=126$, the number of dollars required.

Ex. 3.—Given that y varies as the sum of three quantities, the first of which varies as x^2, the second varies as x, and the third is

constant; and when $x=1, 2, 3$, $y=6, 11, 18$; express y in terms of x.

The three quantities may be represented by mx^2, nx and p.

Then $$y=mx^2+nx+p.$$

Substituting the given values we get

$$6=m+n+p,$$
$$11=4m+2n+p,$$
$$18=9m+3n+p.$$

Solving these equations we get $m=1$, $n=2$, $p=3$.

Therefore $$y=x^2+2x+3, \text{ the result required.}$$

REMARK.—Since $y \propto mx^2+nx+p$, it would at first appear that we should assume

$$y=k(mx^2+nx+p).$$

This equation, however, differs from the previous one only in writing km, kn and kp for m, n and p, so that the *form* of the two equations is the same. The constant k is included in the constants m, n and p of the equations employed.

EXERCISE V.

1. If $y \propto x$, and when $x=5$, $y=7$, what is the value of x when $y=63$?

2. Given that z varies jointly as x and y, and when $x=1$, $y=5$ and $z=15$, find the value of y when $z=150$ and $x=5$.

3. Given that $4x+5y \propto 2x-5y$, and when $x=10$, $y=1$, find the ratio $x:y$.

4. Given that $y \propto px+q$, and when $x=1, 2$, $y=5, 7$, find the ratio $p:q$.

5. Given that z varies directly as x when y is constant, and inversely as y when x is constant, and when $x=15$, $y=1\frac{7}{8}$ and $z=40$, find the value of z when $x^2=5xy-6y^2$.

6. Given that y varies as the sum of two quantities, one of which is constant and the other varies inversely as x, and when $x=1, 7$, $y=12, 6$, find the value of x when $y=10$.

7. Given that $y^2 \propto a^2 - x^2$, and that when $x = 0$, $y = \pm b$, find the value of y when $x = \sqrt{a^2 - b^2}$.

8. Given that $s \propto t^2$ when f is constant and $s \propto f$ when t is constant, and when $t = 1$, $2s = f$, find the equation between s, f and t.

9. The surface of a sphere varies as the square of the radius, and its volume varies as the cube of the radius. Find the radius of a sphere whose volume equals the sum of the volumes of spheres whose radii are 3, 4 and 5 feet respectively, and compare its surface with the sum of the surfaces of the three spheres.

10. If $s^2 \propto vt^3$ and $v^4 \propto st^2$ when f is constant, and $s^3 \propto v^2f$ and $v^5 \propto sf^4$ when t is constant, show that $v^2 \propto fs$ when all vary.

11. Given that y varies as the sum of three quantities, the first of which is constant, the second varies as x, and the third varies as x^2, and that when $x = a$, $2a$ and $3a$, $y = 0$, a and $4a$, find y when $x = (n+1)a$.

12. Given that z varies directly as x and x varies inversely as y, and that when $x = 4$, $y + z = 340$, and when $x = 1$, $y - z = 1275$, for what value of x is $y = z$?

13. If $y \propto x$, then $x - y \propto x + y$ and $x^3 + y^3 \propto xy(ax + by)$.

14. If $y \propto z \propto x$, then $x^3 + y^3 + z^3 \propto xyz \propto (x + y + z)^3$.

15. If $ax + by \propto cx + dy$, then $y \propto x$ and $x^2 + y^2 \propto xy$.

16. If $x + y \propto z$ and $z + x \propto y$, then $x \propto y \propto z$ and

$$xy + yz + zx \propto x^2 + y^2 + z^2.$$

17. If $y^2 \propto zx$ and $z^2 \propto xy$, then $x^2 - yz \propto (xyz)^{\frac{2}{3}}$.

18. If $x \propto y^2$, $y^3 \propto z^4$, $z^5 \propto u^6$ and $u^7 \propto v^4$, then $xyzu \propto v^4$.

19. If $x \propto y + z$, $xz \propto u^2 + y^2$ and $y^2 \propto z(x + u)$, then

$$(x + y)(y + z)(z + u)(u + x) \propto xyzu \propto ax^4 + by^4 + (cz^2 + dzu + eu^2)^2.$$

20. If x, y and z be variable quantities such that $y + z - x$ is constant, and that $(x + y - z)(z + x - y) \propto yz$, then $x + y + z \propto yz$.

21. If $(x+y+z)(x+y-z)(x-y+z)(-x+y+z) \propto x^2y^2$, then either $x^2+y^2 \propto z^2$ or $x^2+y^2-z^2 \propto xy$. Give a geometrical interpretation to this example.

22. A locomotive engine without a train can go 24 miles an hour, and its speed is diminished by a quantity which varies as the square root of the number of cars attached. With four cars its speed is 20 miles an hour. Find the greatest number of cars which the engine can move.

23. If the attendance at church varies directly as the preacher's ability, and inversely as the square root of the length of his sermon; and if 240 and 350 persons attend A's and B's churches when the sermons are 49 and 36 minutes long respectively, compare A's and B's ability.

24. The time of the vibration of a pendulum varies directly as the square root of its length, and inversely as the square root of the force of gravity; and gravity varies inversely as the square of the distance from the earth's centre. Find the height of a tower on whose top a seconds pendulum loses one second in 24 hours. If the length of the pendulum be 39.1393 inches, how much must it be shortened to make it keep correct time in the elevated position?

25. The value of a diamond varies as the square of its weight. Three rings of equal weight, each composed of a diamond set in gold, have values of a, b and c dollars, the diamonds in them weighing 3, 4 and 5 carats respectively. Find the value of a diamond of one carat, the value of the workmanship being the same for each ring.

26. The value of diamonds varies as the square of their weight, and the square of the value of rubies varies as the cube of their weight. A diamond of a carats is worth m times the value of a ruby of b carats, and both together are worth \$$c$. Find the values of a diamond and of a ruby, each weighing n carats.

27. The velocity of a railway train varies directly as the square

root of the quantity of coal used per mile, and inversely as the number of carriages in the train. In a journey of 25 miles in half an hour with 18 carriages 10 cwt. of coal is required; how much coal will be consumed in a journey of 21 miles in 28 minutes with 16 carriages?

28. The consumption of coal by a locomotive varies as the square of the velocity. When the speed is 16 miles an hour the consumption of coal per hour is 2 tons; if the price of coal be \$5 per ton, and the other expenses of the engine be \$5.62½ per hour, find the least cost of a journey of 100 miles.

29. The square of the time of a planet's revolution varies as the cube of its distance from the sun; find the time of a revolution of Venus, assuming that the distances of the Earth and Venus from the sun to be 91¼ millions and 66 millions of miles respectively, and taking our year to be 365 days.

30. The attraction of a planet on its satellite varies directly as the mass (M) of the planet, and inversely as the square of the distance (D); also, the square of the time (T) of a satellite's revolution varies directly as the distance, and inversely as the force of attraction. If m_1, d_1, t_1 and m_2, d_2, t_2 are simultaneous values of M, D, T respectively, prove that

$$\frac{m_1 t_1^2}{m_2 t_2^2} = \frac{d_1^3}{d_2^3}.$$

Hence find the time of revolution of that moon of Jupiter whose distance is to the distance of our moon as 35 : 31, having given that the mass of Jupiter is 343 times that of the Earth, and that the moon's period is 27.32 days.

CHAPTER V.

ARITHMETICAL PROGRESSION.

78. A **Series** is a succession of numbers or quantities which are formed in order according to some definite law.

Thus 1, 2, 3, 4 .... is a series, the law of formation being that each number is obtained from the preceding by adding a unit.

Also, $a+b$, a^2+b^2, a^3+b^3 .... is a series of which the law of formation is evident.

79. An **Arithmetical Series,** or an **Arithmetical Progression,** is a succession of numbers which constantly increase or constantly decrease by a common difference.

Thus each of the following series is an arithmetical progression:

$$1,\ 2,\ 3,\ 4,\ 5\ \ldots.$$
$$20,\ 17,\ 14,\ 11\ \ldots.$$
$$a,\ a+d,\ a+2d,\ a+3d\ \ldots.$$
$$a,\ a-d,\ a-2d,\ a-3d\ \ldots.$$

The words "arithmetical progression" are briefly denoted by the letters A. P.

80. Each of the successive numbers in a series is called a **Term**; the first and the last terms are sometimes called **Extremes,** and the intermediate ones, **Means.** In an arithmetical series the difference between the successive terms is called the **Common Difference.**

81. Any arithmetical series may be represented by

$$a,\ a+d,\ a+2d,\ a+3d\ \ldots.$$

in which a stands for the first term and d the common difference. The series will be an *increasing* or a *decreasing* one according as d is *positive* or *negative.*

82. In any arithmetical series there are five quantities to be considered:

The first term	which is denoted by a.
The last term	" " " l.
The common difference . . .	" " " d.
The number of terms	" " " n.
The sum of all the terms . .	" " " S.

83. *To find any required term in an arithmetical progression, the first term and the common difference being given.*

Forming the terms in succession we have

$$\overset{1^{\text{st}}}{a},\quad \overset{2^{\text{nd}}}{a+d},\quad \overset{3^{\text{rd}}}{a+2d}\ \ldots\ \overset{10^{\text{th}}}{a+9d}\ \ldots\ \overset{n^{\text{th}}}{a+(n-1)d},$$

from which we observe that—

Any term is found by multiplying the common difference by one less than the number of the term, and adding the product to the first term.

This result is briefly expressed in symbols thus:

$$n^{\text{th}} \text{ term} = a + (n-1)d,$$

or
$$l = a + (n-1)d. \tag{1}$$

84. *To find the sum of any required number of terms of an A. P., the first term and the last term being given.*

Write the series first in the natural order, then in the reverse order, and add.

Then $\qquad S = a + (a+d) + (a+2d) + \ldots\ (l-d) + l,$

and $\qquad S = l + (l-d) + (l-2d) + \ldots\ (a+d) + a.$

$$\therefore\ 2S=(a+l)+(a+l)+(a+l)+\ldots\ldots(a+l)+(a+l)$$
$$=(a+l) \text{ repeated } n \text{ times}$$
$$=n(a+l).$$
$$\therefore\ S=\frac{n}{2}(a+l), \text{ the sum required.} \qquad (2)$$

85. If any two terms of an A. P. be given, the series is completely determined. For, suppose the m^{th} and n^{th} terms are p and q,

then
$$a+(m-1)d=p,$$
$$a+(n-1)d=q.$$

From these two equations a and d may be found, and then the series is known.

86. *To find the arithmetical mean between two given extremes.*

Let a and b denote the given extremes and x the required mean, so that a, x, b are in A. P.

Then
$$x-a=b-x,$$
from which
$$x=\frac{a+b}{2}.$$

From the above it is seen that the arithmetical mean of two quantities is half their sum; it corresponds to what is meant in common language by the word "average."

87. *To insert a given number of arithmetical means between two given extremes.*

Let a and b denote the given extremes, m the number of means, and d the common difference of the resulting series. Then, the total number of terms being $m+2$, from the equation
$$l=a+(n-1)d$$
we get
$$b=a+(m+1)d,$$
from which
$$d=\frac{b-a}{m+1}.$$

Therefore the required means are

$$a+\frac{b-a}{m+1},\quad a+\frac{2(b-a)}{m+1},\quad a+\frac{3(b-a)}{m+1}\quad \ldots\ldots\quad a+\frac{m(b-a)}{m+1},$$

or $$\frac{ma+b}{m+1},\quad \frac{(m-1)a+2b}{m+1},\quad \frac{(m-2)a+3b}{m+1}\quad \ldots\ldots\quad \frac{a+mb}{m+1}.$$

88. The equations, $$l=a+(n-1)d, \tag{1}$$

$$S=\frac{n}{2}(a+l), \tag{2}$$

are the fundamental formulæ of arithmetical progression. From them we can find any two of the five quantities involved when the other three are given, and are thus enabled to solve all possible problems in the subject.

By substituting in (2) the value of l from (1)

we get $$S=\frac{n}{2}\{2a+(n-1)d\}, \tag{3}$$

a very useful equation, which, together with (1) and (2), should be carefully memorized.

89. The following examples illustrate some of the best methods of solving problems in arithmetical progression:

Ex. 1.—Given $a=17$, $d=-3$ and $S=55$, find n and l.

Substituting the given values in the equation,

$$S=\frac{n}{2}\{2a+(n-1)d\},$$

we get $$55=\frac{n}{2}\{34-3(n-1)\}.$$

Simplifying, $$3n^2-37n+110=0,$$

from which $$n=5 \text{ or } 7\tfrac{1}{3}.$$

The value $7\frac{1}{3}$ is not admissible, since, from the nature of the problem, n must be a positive integer.

Then
$$\begin{aligned} l &= a + (n-1)d \\ &= 17 + (5-1)(-3) \\ &= 5. \end{aligned}$$

Ex. 2.—The sum of the second and fifth terms of an A. P. is 47, and the sum of the first four terms is 58; find the eleventh term.

With the usual notation we have,

second term $\quad = a + d,$

fifth term $\quad = a + 4d,$

sum of first four terms $\quad = \dfrac{4}{2}\{2a + (4-1)d\}.$

Then from the problem we have,
$$2a + 5d = 47,$$
$$4a + 6d = 58.$$

Solving these equations in the usual way,
$$a = 1 \text{ and } d = 9.$$

Then eleventh term
$$\begin{aligned} &= 1 + (11-1)9 \\ &= 91. \end{aligned}$$

Ex. 3.—If a, b and c are in A. P., then
$$\frac{2}{9}(a+b+c)^3 = a^2(b+c) + b^2(c+a) + c^2(a+b).$$

Since a, b and c are in A. P., $a + c = 2b$.

Then
$$\frac{2}{9}(a+b+c)^3 = \frac{2}{9}(3b)^3 = 6b^3,$$

and
$$\begin{aligned} a^2(b+c) + b^2(c+a) + c^2(a+b) &= (a+b+c)(ab+bc+ca) - 3abc \\ &= 3b\{b(a+c) + ac\} - 3abc \\ &= 6b^3, \end{aligned}$$

which proves the proposition.

EXERCISE VI.

1. In a series whose first term is 5 and common difference 2, find the tenth, fifteenth and one hundredth terms.

2. In a series whose first term is 35 and common difference -3, find the eighth, fifteenth and n^{th} terms.

3. In the series—

(1) $4, 7, 10 \ldots.$, find the n^{th} term.

(2) $8, 5, 2 \ldots.$, find the n^{th} term.

(3) $2\frac{2}{3}, 2\frac{1}{2}, 2\frac{1}{3} \ldots.$, find the ninth and seventeenth terms.

(4) $-23, -18, -13 \ldots.$, find the thirteenth and $(n-1)^{\text{th}}$ terms.

4. In the series, $17, 14, 11 \ldots.$, which term is $-82, -118, -419$?

5. The first term is 17 and the twentieth term is 150; find the common difference and the fortieth term.

6. The third term is 75 and the eleventh term is 131; find the twentieth term.

7. The first term is $2a-3b$ and the second term is $3a-2b$· find the n^{th} term and the $(2n-1)^{\text{th}}$ term. Which term is $(3p+5)a+3pb$?

8. Sum the series—

(1) $1, 2, 3, 4 \ldots.$ to 100 terms.

(2) $8, 3, -2, -7 \ldots.$ to 20 terms.

(3) $\frac{1}{2}, -\frac{3}{4}, -2 \ldots.$ to 24 terms.

(4) $2n-1, 2n-3, 2n-5 \ldots.$ to n terms.

(5) $\frac{6}{\sqrt{3}}, 3\sqrt{3}, \frac{12}{\sqrt{3}} \ldots.$ to 50 terms.

(6) $n-1, n-2, n-3 \ldots.$ to $2n$ terms.

9. Find the arithmetical mean between 33 and 17, 37 and -5, $m+n$ and $m-n$.

10. Insert five arithmetical means between 2 and 20.

11. Insert x arithmetical means between 1 and x^2.

12. If $m-n-1$ arithmetical means be inserted between m^3 and n^3, what is the common difference of the resulting series?

13. The sum of the second and fourth terms of an A. P. is 30, and the sum of the third and fifth is 38; find the first term and the common difference.

14. The eighth term of an A. P. is greater than the fifth by 24, and the sum of the sixth and tenth terms is 100; find the common difference and the n^{th} term.

15. The first term of an A. P. is 1, and the sum of the first twenty terms is 400; find the thirtieth term.

16. The sum of the first five terms of an A. P. is one-third the sum of the next five terms; the tenth term is 19; find the sum of n terms.

17. Find the sum of eleven terms of the A. P. whose sixth term is 10.

18. Find the sum of the whole series formed by inserting m arithmetical means between a and b.

19. Find the $(n+1)^{\text{th}}$ term of an A. P. whose sum to $(2n+1)$ terms is $(2n+1)c$.

20. Show that the sum of the r^{th} term from the beginning and the r^{th} term from the end of any A. P. is constant for all values of r.

21. The sum of n terms of the series, 1, 4, 7 ...., is to the sum of $2n$ terms as $10:41$; find n.

22. The sum of three terms of an A. P. is 33, and the sum of their squares is 413; find the terms.

23. The sum of three terms of an A. P. is 15, and the sum of their cubes is 495; find the terms.

24. Between two numbers whose sum is $2\frac{1}{10}$ a number of means is inserted whose sum is greater by unity than their number; how many means are there?

25. The sum of five terms of an A. P. is 25, and the sum of ten terms is 100; find the sum of n terms.

26. The sum of four numbers in A. P. is 44, and their product is 13440; find the numbers.

27. There are three numbers in A. P.; the square of the first added to the product of the other two is 16; the square of the second added to the product of the other two is 14. Find the numbers.

28. Divide unity into four parts in A. P. such that the sum of their cubes may be $\frac{1}{10}$.

29. The p^{th} term of an A. P. is $\frac{1}{q}$, and the q^{th} term is $\frac{1}{p}$; find the sum of pq terms.

30. Two persons start from the same point and travel in the same direction, the former at the rate of 1, 2, 3, 4, etc., miles during the successive hours; the latter uniformly at 6 miles an hour, but starting two hours after the former; in how many hours after the first person starts will the two be together? Explain the *two* results.

31. In an A. P. the common difference is twice the first term; prove that the sum of n terms divided by the first term is a perfect square.

32. If S denote the sum of the first n natural numbers, then $8S+1$ is the square of an odd number.

33. If S be the sum and d the difference of an A. P. of n terms,

then the difference of the squares of the first and last terms is $\frac{2}{n}(n-1)dS$.

34. The middle term of an arithmetical series of n terms is p, and the n^{th} term is q times the middle term; find the first term and the common difference.

35. Given the first term and the common difference, find n so that the sum of $2n$ terms may be equal to p times the sum of n terms. Examine the case in which $d=2a$ and $p=4$.

36. The series of natural numbers is arranged in groups thus: 1, $2+3$, $4+5+6$, etc.; find the first and the last numbers of the n^{th} group, the sum of the n^{th} group, and the sum of all the groups.

37. The odd numbers are arranged in groups thus: 1, $3+5$, $7+9+11$, etc.; show that the sum of each group is a perfect cube, and the sum of any number of groups beginning with the first is a complete square.

38. Find the sum of n terms of the series,

$$1+(2+3+4)+(5+6+7+8+9)+\dots.$$

39. In any arithmetical series the sum of any two terms, less the first term, is a term of the series; and the difference of any two terms, increased by the first term, is also a term of the series.

40. If the terms of an A. P. be arranged in groups of n terms each, the sums of these groups will form an A. P. whose common difference is n^2 times the common difference of the original series.

41 Prove that the terms of an arithmetical series will still be in A. P. after any of the following operations have been performed upon them:

(1) If the same quantity be added to or subtracted from each.

(2) If the terms of another arithmetical series be added or subtracted in order

(3) If the terms be multiplied or divided by the same factor.

Examine the effect upon the common difference in the several cases.

90. The student should carefully work out all the different cases of Art. 88. This will afford valuable exercise in the solution of literal equations, and render him familiar with the various formulæ of the subject. Some of the results are worthy of special consideration, of which the following is an example:

Given l, d, S, to find n and a.

From the fundamental equations,

$$l = a + (n-1)d \tag{1}$$

$$S = \frac{n}{2}(a + l). \tag{2}$$

Substituting in (2) the value of a obtained from (1) we get

$$2S = n(2l + d - nd),$$

or $$dn^2 - (2l + d)n + 2S = 0,$$

from which $$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8dS}}{2d}.$$

Similarly from the same equations it may be shown that

$$a = \frac{d \pm \sqrt{(2l+d)^2 - 8dS}}{2},$$

or, having found the values of n as above, they may be substituted in either (1) or (2), and then corresponding values of a may be found.

Thus when l, d and S are given we get *two* values each for n and a. The nature of the problem evidently requires n to be a positive integer; if, then, values be given to l, d and S such that both values of n are positive integers, we shall have two series fulfilling the required conditions.

A numerical example will show how this is possible:

Let $$l = 11, \quad d = 2, \quad S = 32,$$

which give $$n = 4 \text{ or } 8, \text{ and } a = 5 \text{ or } -3.$$

We thus obtain the two series,

$$5,\ 7,\ 9,\ 11, \text{ and } -3,\ -1,\ 1,\ 3,\ 5,\ 7,\ 9,\ 11,$$

each of which satisfies the required conditions.

Similarly the double values for n and l, obtained when a, d and S are given, should be examined.

91. We have seen (Art. 88) that of the five quantities, a, l, d, n, S, three must be given in order to determine the remaining two; if, however, the *form* of the n^{th} term, or of the sum of n terms, be given, the whole series is immediately known. We have seen that the n^{th} term is $a+(n-1)d$, and the sum of n terms, $\frac{n}{2}\{2a+(n-1)d\}$. These may be written $(a-d)+dn$ and $\left(a-\frac{d}{2}\right)n+\left(\frac{d}{2}\right)n^2$, which show that the most general forms of these expressions are $p+qn$ and $rn+Sn^2$, where p, q, r and S are constant numbers for any particular series; but n is a variable number, by giving different values to which in succession any particular term, or the sum of any number of terms, may be found.

92. *Given the n^{th} term of an arithmetical series, $p+qn$, to find the common difference and the sum of n terms.*

The successive terms are formed from the general term by substituting for n the values 1, 2, 3 .... n in succession.

Then
$$\begin{aligned} S &= (p+q)+(p+2q)+(p+3q)+\ldots\ldots(p+nq) \\ &= p+p+\ldots\ldots \text{ to } n \text{ terms} + (1+2+3\ldots\ldots n)q \\ &= np+\frac{n(n+1)q}{2}. \end{aligned}$$

The common difference is evidently q, the coefficient of n in the general term.

93. *Given the sum of n terms of an arithmetical series, $pn+qn^2$, to find the n^{th} term and the common difference.*

Let S_n denote the sum of n terms $=pn+qn^2$, then S_{n-1} will denote the sum of $(n-1)$ terms $=p(n-1)+q(n-1)^2$. Now, if

from the sum of n terms we subtract the sum of $n-1$ terms, the remainder must be the n^{th} term.

Then $$n^{\text{th}} \text{ term} = S_n - S_{n-1}$$
$$= pn + qn^2 - \{p(n-1) + q(n-1)^2\}$$
$$= p + q(2n-1).$$

The common difference is $2q$, the coefficient of n in the n^{th} term.

The first term may be found by writing 1 for n either in the expression for the n^{th} term or for the sum of n terms.

94. It will be instructive to give a different solution to the problems of the two preceding Arts.

In every arithmetical series we have

$$n^{\text{th}} \text{ term} = a + (n-1)d.$$

Now, if $$n^{\text{th}} \text{ term} = p + qn$$
$$= (p+q) + (n-1)q,$$

we see that a, or the first term, is $p+q$, and d, or the common difference, is q.

Again, $$S = \frac{n}{2}\{2a + (n-1)d\}.$$

If, then, $$S = pn + qn^2$$
$$= \frac{n}{2}(2p + 2qn)$$
$$= \frac{n}{2}\{2(p+q) + (n-1)2q\},$$

the first term is $p+q$, and the common difference $2q$, as before.

95. The method of the preceding Art. enables us to assign a meaning to negative and fractional values of n obtained as in Ex. 1, Art. 89.

Let $n = -m$ be a negative value so obtained, then $-m$ written for n satisfies the equation.

$$S = \frac{n}{2}\{2a + (n-1)d\},$$

$$\therefore\ S = \frac{-m}{2}\{2a - (m+1)d\}$$

$$= \frac{m}{2}\{-2a + (m+1)d\}$$

$$= \frac{m}{2}\{2(d-a) + (m-1)d\}.$$

This shows that S is the sum of m terms of the series beginning with $d - a$.

Again, $$S = \frac{-m}{2}\{2a - (m+1)d\},$$

$$\therefore\ -S = \frac{m}{2}\{2(a-d) + (m-1)(-d)\}.$$

This result shows that if we begin with the term $a - d$ and count *backwards* m terms the result will be $-S$. Similarly, meanings may be assigned to fractional values of n.

96. *Ex. 1.*—The p^{th} term of an A. P. is P, and the q^{th} term is Q; find the $(p+q)^{\text{th}}$ term.

We have $$p^{\text{th}}\text{ term} = a + (p-1)d = P,$$

and $$q^{\text{th}}\text{ term} = a + (q-1)d = Q.$$

Therefore $$d = \frac{P-Q}{p-q}$$

Now $$(p+q)^{\text{th}}\text{ term} = a + (p+q-1)d$$

$$= a + (p-1)d + qd$$

$$= P + \frac{q(P-Q)}{p-q}$$

$$= \frac{pP - qQ}{p-q}.$$

Ex. 2.—Find the condition that x, y and z may be the p^{th}, q^{th} and r^{th} terms of an A. P.

We have
$$p^{\text{th}}\text{ term} = a + (p-1)d = x,$$
$$q^{\text{th}}\text{ term} = a + (q-1)d = y,$$
$$r^{\text{th}}\text{ term} = a + (r-1)d = z.$$

From these three equations we must eliminate a and d. Subtracting the equations from each other in succession we get
$$(p-q)d = x - y,$$
$$(q-r)d = y - z,$$
$$(r-p)d = z - x.$$

Now multiply the equations respectively by r, p and q, and add, then the left side vanishes and we get
$$(x-y)r + (y-z)p + (z-x)q = 0,$$
the condition required. Had we multiplied by z, x and y instead of r, p and q we should have obtained
$$(p-q)z + (q-r)x + (r-p)y = 0,$$
the same result under a different form.

Ex. 3.—If S_n denote the sum of n terms of an A. P., prove $S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n = 0$.
$$\begin{aligned} S_{n+3} - S_n &= (n+1)^{\text{th}} + (n+2)^{\text{th}} + (n+3)^{\text{th}}\text{ terms} \\ &= 3(n+2)^{\text{th}}\text{ term} \\ &= 3(S_{n+2} - S_{n+1}), \end{aligned}$$
which proves the proposition.

Ex. 4.—If the sum of m terms of an A. P. be equal to the sum of the next n terms, and also equal to the sum of the following p terms, prove
$$\frac{(m+n)}{(n+p)} = \frac{p(m-n)}{m(n-p)}.$$

Equating the sums of the three sets of terms we get

$$\frac{m}{2}\{2a+(m-1)d\}=\frac{n}{2}\{2(a+md)+(n-1)d\}$$

$$=\frac{p}{2}\{2(a+\overline{m+n}\,.\,d)+(p-1)d\}.$$

Therefore $$\frac{2a+(m-1)d}{2(a+md)+(n-1)d}=\frac{n}{m}$$

and $$\frac{2(a+\overline{m+n}\,.\,d)+(p-1)d}{2(a+md)+(n-1)d}=\frac{n}{p}.$$

From which $$\frac{(m+n)d}{2(a+md)+(n-1)d}=\frac{m-n}{m}$$

and $$\frac{(n+p)d}{2(a+md)+(n-1)d}=\frac{n-p}{p}.$$

Then by division $$\frac{m+n}{n+p}=\frac{p(m-n)}{m(n-p)}.$$

EXERCISE VII.

1. How many terms of the series, 25, 23, 21 ...., must be taken that the sum may be 165?

2. The n^{th} term of an A. P. is $2n-3$; find the common difference and the sum of n terms.

3. The r^{th} term of an A. P. is $7-\frac{r}{2}$; find the sum of $2n+1$ terms.

4. The $(n+1)^{\text{th}}$ term of an A. P. is $\frac{ma-nb}{a-b}$; find the sum of $2n+1$ terms.

5. The n^{th} term of a series is n^2-n+1; write down the tenth term, the r^{th} term, the $(n+1)^{\text{th}}$ term. Is it an arithmetical series?

6. The sum of n terms of an A. P. is $5n^2-3n$; find the n^{th} term and the common difference.

7. The sum of n terms of the series, 2, 5, 8 . . . . , is 950; find n. Give an interpretation to the negative value of n.

8. Find the eighth term of an A. P. whose sum to n terms is $\frac{n}{2}\left\{\frac{n}{3}+\frac{1}{4}\right\}$.

9. The n^{th} term of an A. P. is $2n+1$; of how many terms is the sum 99? Give two different interpretations to the negative result.

10. If the sum of p terms of an A. P. equals the sum of q terms, then the sum of $p+q$ terms is zero.

11. Find the relation between a and d when the equation between a, d, s, n, is satisfied by two different positive integral values of n.

12. The sum of n terms of an A. P. is an^2+bn, and the sum of m terms of another series is bm^2+am; the fifth term of the first series is half the fifth term of the second series; show that $17a=7b$.

13. The sums of two arithmetical series, each to n terms, are to each other as $13-7n$ to $1+3n$; find the ratios of their first terms, their second terms, and their r^{th} terms.

14. If a, b and c are the p^{th}, q^{th} and r^{th} terms of an A. P., then p, q and r are the a^{th}, b^{th} and c^{th} terms of another A. P.; a, b and c being positive integers.

15. If the $(p+q)^{\text{th}}$ and $(p-q)^{\text{th}}$ terms of an A. P. be m and n respectively, find the p^{th} and q^{th} terms.

16. Divide $\frac{n(n+3)}{12}$ into n parts such that each shall exceed the preceding by a fixed quantity.

17. The sum of m terms of an A. P. is n, and the sum of n terms is m; show that the sum of $m+n$ terms is $-(m+n)$, and the sum of $m-n$ terms is $(m-n)\left(1+\frac{2n}{m}\right)$.

18. If S_n denote the sum to n terms of an A. P., show that $S_{n+2}-2S_{n+1}+S_n$ is equal to the common difference.

19. If S_a denote the sum of n terms of the natural numbers beginning with a, prove $S_{3a+n-1}=3S_a$.

20. If a_1, a_2, $a_3 \dots\dots a_n$ denote the terms of an A. P., and S their sum what is the sum of the series,

$$(S-a_1)+(S-a_2)+(S-a_3)+\dots\dots(S-a_n)?$$

21. There are n arithmetical series of n terms each; their first terms are the natural numbers, 1, 2, 3 ...., and the common difference of each is the same as the first term· find the sum of all the terms of the series taken together.

22. If S and S' denote the sum to n terms of two arithmetical series having the same first term a, and common differences d and $-d$ respectively, then $\dfrac{S-S'}{S+S'}=\dfrac{(n-1)d}{2a}$.

23. If S_n denote the sum of n terms of an arithmetical series, then
$$S_m-S_n : S_{m+n}=m-n : m+n$$
and
$$S_{3n}-3S_{2n}+3S_n=0.$$

24. If the sum of the first m terms of an A. P. equals the sum of n terms beginning with the $(r+1)^{\text{th}}$, and also equals the sum of p terms beginning with the $(s+1)^{\text{th}}$, prove

$$\frac{2r+n-m}{2s+p-m}=\frac{(m-n)p}{(m-p)n}.$$

25. If the p^{th} and q^{th} terms of an A. P. are P and Q respectively, what is the n^{th} term?

26. If S be the sum of n terms of an A. P., and S' the sum of the arithmetical means between the consecutive terms, then

$$S : S'=n : n-1.$$

27. If $a_1+a_2+a_3+\dots\dots a_n=\dfrac{nS}{2}$, show that

$$(S-a_1)^2+(S-a_2)^2+\dots\dots(S-a_n)^2=a_1^2+a_2^2+\dots\dots a_n^2.$$

28. The sides of a right-angled triangle are in A. P.; show that they are proportional to 3, 4, 5.

29. If a^2, b^2, c^2 are in A. P., then $\frac{1}{b+c}$, $\frac{1}{c+a}$, $\frac{1}{a+b}$ are also in A. P.

30. If a, b, c are in A. P., then $a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ are in A. P.; also, a^2-bc, b^2-ca, c^2-ab are in A. P. Compare the common differences in the several cases.

31. If $\frac{a+b}{1-ab}$, b, $\frac{b+c}{1-bc}$ are in A. P., then $\frac{1}{a}$, b, $\frac{1}{c}$ are in A. P.

32. If $\frac{a}{b-c}$, $\frac{b}{c-a}$, $\frac{c}{a-b}$ are in A. P., then

$$\frac{a^3+c^3-2b^3}{a^2+c^2-2b^2}=\frac{a+b+c}{2}.$$

33. If two terms of one A. P. are proportional to the corresponding terms of another A. P., then all the terms of the former series are proportional to the corresponding terms of the latter series.

34. If S_n denote the sum of n terms of an A. P., then

$$S_{n+2p}-2S_{n+p}+S_n=p^2d,$$
$$S_{n+3p}-3S_{n+2p}+3S_{n+p}-S_n=0.$$

35. Find an A. P. beginning with unity in which the sum of the first half of any even number of terms bears a constant ratio to the sum of the second half, and show that there is but one such series.

36. There are a number of series, each in A. P., whose common differences are 1, 2, 3 . . . . , and whose sums, each to n terms, are n^2; show that their first terms form a decreasing A. P., whose first term is $\frac{1}{2}(n+1)$, and common difference $\frac{1}{2}(n-1)$.

37. If S_1, S_2. . . . be the sums of m arithmetical series, each

to n terms, the first terms being 1, 2, 3 ...., and the differences 1, 3, 5 ....; show that $S_1 + S_2 + \ldots . S_m = \frac{1}{2} mn(mn+1)$.

38. If S_1 denote the sum of n terms of the series 1, 5, 9 ...., and S_2 the sum of $n-1$ terms, or of n terms, of the series 3, 7, 11 ...., then $S_1 + S_2 = (S_1 - S_2)^2$.

39. Find the first of n consecutive odd numbers whose sum is n^p, where p is any positive integer greater than unity.

40. Show that n^p may be resolved into the difference of two integral squares, n and p being integral, and p greater than 2.

41. If S_r denote the sum of n terms of the natural numbers beginning with r, then $S_1 + S_2 + \ldots . S_m = \frac{mn(m+n)}{2}$.

42. The successive terms of an A. P. are arranged in groups of 1, 2, 3, etc., terms each; show that if S_r denote the r^{th} group,

then
$$S_n = na + \frac{n}{2}(n^2-1)d,$$

and
$$S_1 + S_2 + \ldots . S_n = \frac{n(n+1)}{8}\{4a + (n-1)(n+2)d\}.$$

43. With the notation of the preceding example show that

$$(m-n)\{S_{m+n} - S_m - S_n\} + (m+n)\{S_{m-n} - S_m + S_n\} = 0,$$

and $p(S_m - S_n) + m(S_n - S_p) + n(S_p - S_m)$
$$= (m-n)(n-p)(p-m)(m+n+p)\frac{d}{2}.$$

CHAPTER VI.

GEOMETRICAL PROGRESSION.

97. A **Geometrical Series,** or a **Geometrical Progression,** is a succession of numbers which constantly increase or constantly decrease by a common factor.

Thus each of the following series is a geometrical progression:

$$1,\ 2,\ 4,\ 8,\ 16\ \ldots.$$

$$72,\ -36,\ 18,\ -9,\ 4\tfrac{1}{2}\ \ldots.$$

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4\ \ldots.$$

The words "geometrical progression" are briefly denoted by the letters G. P., and the common factor is frequently called the **Common Ratio,** or briefly, the **Ratio.**

98. Any geometrical series may be represented by

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4\ \ldots.$$

in which a represents the first term and r the common ratio. The series will increase or decrease numerically according as r is greater or less than unity.

99. In any geometrical series there are five quantities to be considered, viz.:

The first term	which is denoted by a.
The common ratio	" " " r.
The last term	" " " l.
The number of terms	" " " n.
The sum of all the terms. . .	" " " S.

100. *To find any required term in a geometrical progression, the first term and the common ratio being given.*

Forming the terms in succession we have

$$\begin{array}{ccccccc} 1^{\text{st}} & 2^{\text{nd}} & 3^{\text{rd}} & & 10^{\text{th}} & & n^{\text{th}} \\ a, & ar, & ar^2 & \ldots\ldots & ar^9 & \ldots\ldots & ar^{n-1}, \end{array}$$

from which we observe that—

Any term is found by multiplying the first term by the common factor raised to a power less by one than the number of the term.

This result is briefly expressed in symbols thus:

$$n^{\text{th}} \text{ term} = ar^{n-1},$$

or $$l = ar^{n-1}. \qquad (1)$$

101. *To find the sum of a given number of terms of a geometrical series, the first term and the common ratio being known.*

Let n be the number of terms required, and denote the sum by S.

Then $$S = a + ar + ar^2 + ar^3 + \ldots\ldots ar^{n-2} + ar^{n-1},$$

therefore $$rS = \quad ar + ar^2 + ar^3 + \ldots\ldots ar^{n-2} + ar^{n-1} + ar^n.$$

Then by subtraction $$rS - S = ar^n - a,$$

therefore $$S = \frac{a(r^n - 1)}{r - 1}. \qquad (2)$$

Again, from (1) $$rl = ar^n.$$

substituting in (2) we get $$S = \frac{rl - a}{r - 1}. \qquad (3)$$

Equations (1), (2) and (3) should be committed to memory.

102. If any two terms of a G. P. be given, the series is completely determined. For, suppose the m^{th} and n^{th} terms are p and q,

then $$ar^{m-1} = p$$

and $$ar^{n-1} = q.$$

From these two equations a and r may be found, and then the series is known.

103. *To find the geometric mean between two given extremes.*

Let a and b denote the given extremes and x the required mean, so that a, x, b are in G. P.

Then $$\frac{x}{a} = \frac{b}{x},$$

from which $$x = \sqrt{ab}.$$

From the above it is seen that the geometric mean between two quantities is *the square root of their product;* it is the same as "mean proportional." This arises from the fact that a geometric series is simply a series of numbers in continued proportion.

104. *To insert a given number of geometric means between two given extremes.*

Let a and b be the given extremes, m the number of means, and r the common ratio. Then since the total number of terms is $m+2$, of which a is the first and b the last, we get from (1)

$$b = ar^{m+1} \text{ or } r = \left(\frac{b}{a}\right)^{\frac{1}{m+1}}.$$

Therefore the means required are

$$a\left(\frac{b}{a}\right)^{\frac{1}{m+1}}, \quad a\left(\frac{b}{a}\right)^{\frac{2}{m+1}}, \quad a\left(\frac{b}{a}\right)^{\frac{3}{m+1}} \dots\dots a\left(\frac{b}{a}\right)^{\frac{m}{m+1}},$$

or $$(a^m b)^{\frac{1}{m+1}}, \quad (a^{m-1}b^2)^{\frac{1}{m+1}}, \quad (a^{m-2}b^3)^{\frac{1}{m+1}} \dots\dots (ab^m)^{\frac{1}{m+1}}.$$

This series should be compared with that of Art. 87, when it will appear that *addition* has been changed to *multiplication*, *coefficients* to *exponents*, and *division* to *a root sign.* These changes all arise from the fact that the successive terms in A. P. are formed by *addition*, while those of G. P. are formed by *multiplication.*

105. The equations, $l = ar^{n-1}$,

$$S = \frac{a(r^n - 1)}{r - 1},$$

are the fundamental formulæ of geometrical progression. When any three of the five quantities involved are given, they theoretically determine the other two. If, however, we attempt to work out the different cases in succession we shall find that some of the equations which present themselves cannot be solved by the methods already given, and that others cannot be solved by any known method whatever.

Four cases present no difficulty, viz.: when (1) a, r, n, (2) a, n, l, (3) r, n, l, (4) r, n, s, are given.

The four cases in which n has to be found require logarithms for their solution.

The remaining cases, viz.: when (1) a, n, s, (2) n, l, s, are given, are incapable of a general solution.

EXERCISE VIII.

Find the required terms in the following series:

1. The fifth term of 2, 4, 8, 16....
2. The tenth term of 1, 3, 9, 27....
3. The fifteenth term of 64, 32, 16, 8....
4. The n^{th} term of 2, 6, 18, 54....
5. The $(2n-1)^{\text{th}}$ term of 4, -8, 16, -32....
6. The n^{th} term of a, a^2r^2, a^3r^4, a^4r^6....
7. The $2n^{\text{th}}$ term of $\frac{a^2}{b}$, $-a$, b, $-\frac{b^2}{a}$....
8. The $(n-1)^{\text{th}}$ term of $3a$, $5a^2r$, $7a^3r^2$, $9a^4r^3$....
9. The n^{th} term of $\frac{\sqrt{3}}{\sqrt{3}+1}$, $\frac{\sqrt{3}}{\sqrt{3}+2}$, $\frac{3-\sqrt{3}}{\sqrt{3}+2}$....
10. The $(2n+3)^{\text{th}}$ term of $\frac{\sqrt{2}+1}{\sqrt{2}-1}$, $\frac{1}{2-\sqrt{2}}$, $\frac{1}{2}$....

11. Sum $1+2+4+8 \ldots$ to ten terms and to n terms.

12. Sum $\frac{8}{5}+\frac{8}{3}+\frac{40}{9} \ldots$ to six terms and to n terms.

13. Sum $1-2+4-8 \ldots$ to $2n$ terms and to $2n+1$ terms.

14. Sum $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} \ldots$ to ten terms and to n terms.

15. Sum $a^2x^{-1}-a^3+a^4x-a^5x^2 \ldots$ to n terms and to $2n-3$ terms.

16. Sum $(2-\sqrt{3})+1+(2+\sqrt{3}) \ldots$ to $n+1$ terms.

17. Sum to n terms the series whose n^{th} term is ar^{n-3}.

18. Sum to $n+3$ terms the series whose n^{th} term is $a^{1-n}b^{2n-1}$.

19. Sum $\frac{1}{3}+\frac{1}{2}+\frac{3}{4}+\ldots$ to $n+1$ terms.

20. Sum $2+\sqrt[4]{8}+\sqrt{2}+\ldots$ to n terms and to 16 terms.

21. Sum $\frac{5}{6}-\frac{5}{9}+\frac{10}{27}-\ldots$ to n terms and to $2n+1$ terms.

22. The first term of a geometrical series is 5 and the third term is 80; find the common ratio.

23. The fifth term of a geometrical series is 48, and the ratio is 2; find the first and n^{th} terms.

24. If $a=2$ and $r=3$, which term will be 162?

25. The sum of three terms of a geometrical series is $14\frac{1}{3}$, and their product is 8; find the terms.

26. The sum of three numbers in G. P. is 13, and the sum of their squares is 91; find the numbers.

27. The sum of two numbers is m, and their geometric mean is n; find the numbers.

28. Insert two geometrical means between 9 and $21\frac{1}{3}$.

29. Insert six geometrical means between $\frac{8}{27}$ and $5\frac{1}{16}$.

30. What is the common ratio of a geometrical series when

the difference between the first and n^{th} terms is equal to the sum of $n-1$ terms?

31. The sum of the first three terms of a geometrical series is $4\frac{3}{4}$, and the sum of the first, third and fifth terms is $8\frac{5}{16}$; find the series.

32. The sum of the first six terms of a geometrical series is $157\frac{1}{2}$, and the sum of the third to the eighth inclusive is 630; find the series.

33. The first of four numbers in G. P. is $\dfrac{1}{17}$, and their sum is greater by one than the common ratio; find the numbers.

34. There are five numbers, the first three of which are in G. P., and the last three in A. P., the second number being the common difference of these three terms. The sum of the last four is 40, and the product of the second and last is 64; find the numbers.

35. Three numbers whose sum is 27 are in A. P.; if 1, 3 and 11 be added to them respectively the results will be in G. P.; find the numbers.

36. To each of the first two of the numbers, 3, 35, 190, 990, is added x, and to each of the last two is added y; the resulting numbers are in G. P.; find x and y.

37. The series of natural numbers are divided into groups thus: $1,\ 2+3,\ 4+5+6+7$, etc., each group containing twice as many numbers as the preceding; find the sum of the n^{th} group and the sum of all the groups.

38. The odd numbers are divided into groups thus: $1,\ 3+5,\ 7+9+11+13$, etc., each group containing twice as many numbers as the preceding; find the last number of the n^{th} group, the sum of the n^{th} group, and the sum of all the groups.

39. The terms of the geometrical series, $1,\ 2,\ 4,\ 8 \ldots.$, are arranged in groups thus: $1,\ 2+4,\ 8+16+32$, etc., each group

containing one number more than the preceding; find the sum of the n^{th} group and the sum of n groups.

40. The terms of the geometrical series, 1, 2, 4, 8 . . . . , are arranged in groups thus: 1, $2+4$, $8+16+32+64$, etc., each group containing twice as many terms as the preceding; find the sum of all the groups and the sum of n^{th} group.

41. If a geometric progression consist of $4n$ terms, show that the ratio of the sum of the first n terms and the last n terms is to the sum of the remaining $2n$ terms as $r^{2n}-r^n+1$ to r^n.

42. Find the sum of the squares of the differences of every two consecutive terms in a G. P. of $n+1$ terms.

43. Determine m and n in terms of a and b so that $\dfrac{ma+nb}{m+n}$ may be the arithmetical mean between m and n, and the geometrical mean between a and b.

44. In a G. P. with the usual notation prove

$$aS_{2n}=S_n(S_{n+1}-r\,.\,S_{n-1}) \text{ and } S_n(S_{3n}-S_{2n})=(S_{2n}-S_n)^2.$$

45. If P be the continued product of n quantities in G. P., S their sum, and S' the sum of their reciprocals, show that $P^2=\left(\dfrac{S}{S'}\right)^n$.

46. If S_n denotes the sum of n terms of a G. P., S_{2n} the sum of the next $2n$ terms, and S_{3n} the sum of the following $3n$ terms, then $r^n\,.\,S_{2n}{}^2=S_n\,.\,S_{3n}+r^{4n}\,.\,S_n{}^2$.

47. If $(a_1{}^2+a_2{}^2+a_3{}^2+\ldots\ldots a_{n-1}{}^2)(a_2{}^2+a_3{}^2+a_4{}^2+\ldots\ldots a_n{}^2)$
$$=(a_1a_2+a_2a_3+\ldots\ldots a_{n-1}a_n)^2,$$
and $a_1, a_2 \ldots\ldots a_n$ are all real, then $a_1, a_2, \ldots\ldots a_n$ are in G. P.

106. The sum of n terms of a geometrical series has been shown to be $\dfrac{a(r^n-1)}{r-1}$; this may be written $\dfrac{a(1-r^n)}{1-r}$. Now if r be less than unity, by taking n large enough r^n may be made as small

as we please (Art. 15), *i.e.*, the sum of n terms may be made as nearly equal as we please to $\frac{a}{1-r}$. This statement is usually abbreviated thus:

The sum of the series, $a + ar + ar^2 + \ldots\ldots$ *ad infinitum*, is $\frac{a}{1-r}$.

The same result may be obtained thus:

Let $$S = a + ar + ar^2 + ar^3 + \ldots\ldots$$

then $$rS = \quad ar + ar^2 + ar^3 + \ldots\ldots$$

$$\therefore\ S(1-r) = a, \text{ or } S = \frac{a}{1-r} \text{ as before.}$$

This process deserves very careful attention. Since the series in each line is continued indefinitely it is assumed that the terms cancel each other entirely. In reality, however, *one term* is neglected, which corresponds with assuming r^n to be zero in the former investigation; this is legitimate only when the term omitted is *indefinitely small.* The necessity for care in such matters will easily be seen from the following:

Let $$S = 1 + 2 + 4 + 8 + \ldots\ldots$$

then $$2S = \quad 2 + 4 + 8 + \ldots\ldots$$

$$\therefore\ S = -1.$$

This absurdity arises from the fact that the single term neglected is more important than the whole of those retained.

107. Recurring decimals in arithmetic are familiar examples of infinite geometrical series.

Thus $$\cdot\dot{7} = \cdot 777\ldots\ldots = \frac{7}{10} + \frac{7}{10}\left(\frac{1}{10}\right) + \frac{7}{10}\left(\frac{1}{10}\right)^2 + \ldots\ldots$$

which is an infinite geometrical series, whose first term is $\frac{7}{10}$ and common ratio $\frac{1}{10}$; its sum to infinity is therefore

$$\frac{7}{10} \div \left(1 - \frac{1}{10}\right) = \frac{7}{9}.$$

Again,
$$\cdot 2\dot{3}\dot{4} = \frac{2}{10} + \frac{34}{10^3} + \frac{34}{10^5} + \frac{34}{10^7} + \dots$$
$$= \frac{2}{10} + \frac{34}{10^3}\left\{1 + \frac{1}{10^2} + \frac{1}{10^4} + \dots\right\}$$
$$= \frac{2}{10} + \frac{34}{10^3} \cdot \frac{1}{1 - \frac{1}{10^2}}$$
$$= \frac{2}{10} + \frac{34}{990}$$
$$= \frac{232}{990}.$$

The value of any recurring decimal may be found in the same way, but the result may be more neatly obtained by the method of Art. 106; the solution is given below.

108. *To find the value of a recurring decimal.*

Let P denote the figures which do not recur, and let them be p in number; let Q denote the recurring period consisting of q digits; let D denote the value of the whole decimal.

Then $\qquad D = \cdot PQQQ \dots$

Therefore $\qquad 10^p \times D = P\cdot QQQQ \dots$

and $\qquad 10^{p+q} \times D = PQ\cdot QQQQ \dots$

Therefore $\qquad (10^{p+q} - 10^p)D = PQ - P,$

or
$$D = \frac{PQ - P}{(10^q - 1)10^p}.$$

Now, $10^q - 1$ is a number consisting of q nines, and 10^p is a unit followed by p ciphers.

From this result the truth of the ordinary rule for reducing a recurring decimal to its equivalent vulgar is at once evident.

109. To find the value of nr^n when r is less than unity and n is indefinitely great.

Since r is less than unity, $1-r$ is positive, and a number, x, may be taken large enough so that

$$\frac{r}{x} < 1-r, \text{ or } \left(1+\frac{1}{x}\right)r < 1.$$

Put $$\left(1+\frac{1}{x}\right)r = m, \text{ so that } m < 1,$$

then $$\left(1+\frac{2}{x}\right)r^2 < m^2,$$

$$\left(1+\frac{3}{x}\right)r^3 < m^3$$

$$\cdots\cdots$$

$$\left(1+\frac{n}{x}\right)r^n < m^n,$$

or $$(x+n)r^n < x\,.\,m^n.$$

Therefore $$nr^n < x\,.\,m^n.$$

Now, m is less than unity, therefore when n is indefinitely great, m^n is indefinitely small; and since x is a finite number, xm^n is also indefinitely small; and $nr^n < xm^n$, therefore when r is less than unity, by taking n large enough nr^n may be made less than any finite quantity.

110. To find the sum of n terms of the series,

$$a+(a+d)r+(a+2d)r^2+\ldots\ldots\{a+(n-1)d\}r^{n-1},$$

in which each term is the product of the corresponding terms of an arithmetic and a geometric series.

Denote the sum by S, then

$$S = a+(a+d)r+(a+2d)r^2+\ldots\{a+(n-1)d\}r^{n-1},$$

$$rS = \qquad ar+(a+d)r^2+\ldots\{a+(n-2)d\}r^{n-1}+\{a+(n-1)d\}r^n.$$

$$\therefore S(1-r) = a + dr + dr^2 + \ldots dr^{n-1} - \{a+(n-1)d\}r^n$$

$$= a + \frac{dr(1-r^{n-1})}{1-r} - \{a+(n-1)d\}r^n,$$

or $$S = \frac{a - \{a+(n-1)d\}r^n}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2}.$$

The above series is sometimes called an arithmetico-geometric series.

The last term may be written $ar^{n-1} + d(n-1)r^{n-1}$. If r be less than unity, by taking n large enough this term may be made indefinitely small (Art. 109), and may therefore be neglected. Omitting this term and summing the series $dr + dr^2 + \ldots$ to infinity we get $\frac{a}{1-r} + \frac{dr}{(1-r)^2}$ as the sum of the preceding series to infinity.

EXERCISE IX.

Sum the following series *ad infinitum:*

1. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots.$

2. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots.$

3. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots.$

4. $\frac{3}{4} - \frac{3}{8} + \frac{3}{16} - \ldots.$

5. $\frac{2}{3} - \frac{1}{2} + \frac{3}{8} - \ldots.$

6. $2 + \sqrt{2} + 1 + \ldots.$

7. $2 - \sqrt[3]{4} + \sqrt[3]{2} - \ldots.$

8. $(2+\sqrt{3}) + 1 + (2-\sqrt{3}) \ldots.$

9. $\frac{1}{\sqrt{2}} - \frac{1}{3} + \frac{\sqrt{2}}{9} - \frac{2}{27} + \ldots$

10. $a + \sqrt{\frac{a}{b}} + \frac{1}{b} + \ldots.$

11. $\frac{\sqrt{2}+1}{\sqrt{2}-1}, \frac{1}{2-\sqrt{2}}, \frac{1}{2} \ldots.$

12. $\frac{\sqrt{3}}{\sqrt{3}+1}, \frac{\sqrt{3}}{\sqrt{3}+2}, \frac{3-\sqrt{3}}{\sqrt{3}+2} \ldots$

13. Sum $1 + 2x + 3x^2 + 4x^3 + \ldots$ to n terms and *ad inf.*, x being less than unity.

14. Sum $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\ldots$ to n terms and *ad inf.*

15. Sum $1+\frac{3}{2}+\frac{5}{2^2}+\frac{7}{2^3}+\ldots$ to n terms and *ad inf.*

16. Sum $1-\frac{3}{2}+\frac{5}{2^2}-\frac{7}{2^3}+\ldots$ to n terms and *ad inf.*

17. Sum $ar+(a+ab)r^2+(a+ab+ab^2)r^3+\ldots$ to n terms and *ad inf.*, r and br being each less than unity.

18. In a geometrical series consisting of an odd number of terms prove that the product of the first and last equals the square of the middle term.

19. The $(n+1)^{\text{th}}$ term of a geometrical series is c; find the product of $2n+1$ terms.

20. If n geometrical means be inserted between a and b, what is the product of the terms of the whole series thus formed?

21. In a G. P. the $(p+q)^{\text{th}}$ term is m, and the $(p-q)^{\text{th}}$ term is n; find the p^{th} and the q^{th} terms.

22. If the p^{th} term in a G. P. is P, and the q^{th} term is Q, what is the n^{th} term?

23. If a, b and c be the p^{th}, q^{th} and r^{th} terms of a geometrical progression, then $a^{q-r}\,.\,b^{r-p}\,.\,c^{p-q}=1$.

24. If the p^{th}, q^{th}, r^{th} and s^{th} terms of an A. P. are in G. P., then $p-q$, $q-r$ and $r-s$ are also in G. P.

25. In a G. P., if each term be added to, or subtracted from, the preceding, the results in either case will be in G. P.

26. If the terms of a geometrical series be arranged in groups of p terms each, the sums of the successive groups will be in G. P. Find the sum of n groups, and show that it is equal to the sum of the same terms taken separately.

27. If S be the sum of an odd number of terms in G. P., and

if S' be the sum of the series when the signs of the even terms are changed, show that the sum of the squares of the terms will be SS'.

28. If there be any number of quantities in G. P., r the common ratio and S_n the sum of the first n terms, prove that the sum of the products of every two terms is

$$\frac{r}{r+1} \cdot S_n \cdot S_{n-1}.$$

29. If P be the sum of the series, $1 + r^p + r^{2p} + r^{3p} + \ldots$ *ad inf.*, and if Q be the sum of the series, $1 + r^q + r^{2q} + r^{3q} + \ldots$ *ad inf.*, show that $P^q(Q-1)^p = Q^p(P-1)^q$.

30. Sum the series, $\sqrt{2} + \frac{2}{3}\sqrt{3} + \frac{2}{3}\sqrt{2} + \ldots$ *ad inf.*

31. Sum the series,

$$\frac{1}{\sqrt{2}(1+\sqrt{2})} + \frac{1}{(1+\sqrt{2})(2+\sqrt{2})} + \frac{1}{(2+\sqrt{2})(3+\sqrt{2})} + \ldots \textit{ ad inf.}$$

32. Sum to n terms $3 + 33 + 333 + \ldots$

33. Find the series in which each term equals n times the sum of all which follow it, the sum of the first two terms being m.

34. If S_n be the sum of n terms of a G. P., what are the sums of $S_1 + S_2 + \ldots S_n$ and $S_n + S_{n+p} + S_{n+2p} + \ldots S_{n+(m-1)p}$.

35. Show that $2^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 8^{\frac{1}{8}} \cdot 16^{\frac{1}{16}} \ldots$ *ad inf.* $= 4$,

and that $3^{\frac{1}{3}} \cdot 9^{\frac{1}{9}} \cdot 27^{\frac{1}{27}} \cdot 81^{\frac{1}{81}} \ldots$ *ad inf.* $= 3^{\frac{3}{4}}$.

36. If $S_1, S_2, S_3 \ldots S_n$ be the sums of n terms of n geometrical progressions, of which all the first terms are 1 and common ratios 1, 2, 3 …. n respectively, show that

$$S_1 + S_2 + 2S_3 + 3S_4 + \ldots (n-1)S_n = 1^n + 2^n + 3^n + \ldots n^n.$$

37. If S_n denote the sum of the n^{th} powers of the terms of an infinite geometrical series, show that

$$\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \ldots \textit{ ad inf.} = \frac{1}{a-1} - \frac{r}{a-r}.$$

38. Given S the sum and s^2 the sum of the squares of the terms of an infinite G. P., show that the sum of the series to n terms is

$$S\left\{1-\left(\frac{S^2-s^2}{S^2+s^2}\right)^n\right\}.$$

39. The middle points of the sides of a triangle are joined, the middle points of the sides of the triangle so formed are again joined, and so on *ad infinitum*. Show that the sum of the areas of all the triangles so formed is one-third the area of the original triangle.

40. Two straight lines meet forming an acute angle; from any point in one a perpendicular is drawn to the other; from the foot of this perpendicular a perpendicular is drawn to the former, and so on *ad infinitum*. The lengths of the first two perpendiculars are a and b; find the sum of the lengths of all the perpendiculars and the sum of the areas of all the right-angled triangles thus formed.

41. The triangle ABC has each of the angles at B and C double the angle at A; lines are drawn within the triangle, making the triangles CDB, DEB, etc., each similar to the original triangle. If A denote the area of the original triangle, find the sum of the areas of the infinite series of triangles, ABC, CDB, DEB, etc., and also of the infinite series, CDA, DEC, etc.

CHAPTER VII.

HARMONICAL PROGRESSION.

111. An **Harmonical Series,** or an **Harmonical Progression,** is a series of numbers such that of every three consecutive terms the ratio of the first to the third equals the ratio of the difference between the first and second to the difference between the second and third, the differences being always taken in the same order.

Thus a, b, c, d.... are in harmonical progression

if
$$a : c = a - b : b - c,$$
$$b : d = b - c : c - d, \text{ etc.}$$

The numbers 30, 20, 15, 12, 10.... are in harmonical progression,

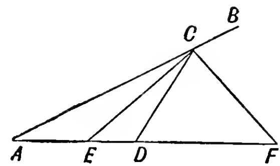
for
$$30 : 15 = 30 - 20 : 20 - 15,$$
$$20 : 12 = 20 - 15 : 15 - 12, \text{ etc.}$$

The words "harmonical progression" are briefly denoted by the letters H. P.

112. Harmonical Progression, formerly called Musical Progression, derives its name from the fact that musical strings of uniform thickness and tension produce *harmony* when their lengths are in progression according to the preceding definition. Its importance is chiefly due to this fact and to the occurrence of harmonical quantities in connection with many geometrical problems.

113. A good example of quantities in H. P. may be obtained as follows:

Take any two straight lines AB and CD cutting each other in C; bisect the angles ACD, BCD by the straight lines CE, CF; across the four lines CA, CE, CD, CF draw any straight line $AEDF$, then the lengths of the lines AE, AD, AF are in harmonical progression.



For $\qquad AE : ED = AC : CD \qquad$ Euc. VI., 3

$\qquad = AF : FD. \qquad$ Euc. VI., A

Therefore $\qquad AE : AF = ED : DF \qquad$ Art. 52, (2)

$\qquad = AD - AE : AF - AD,$

which shows that AE, AD, AF satisfy the conditions for H. P. according to the definition.

114. *If a series of numbers are in H. P., their reciprocals are in A. P., and conversely.*

Let a, b, c be in H. P.

Then, by definition, $\qquad a : c = a - b : b - c. \qquad$ Art. 111

Therefore $\qquad a(b-c) = (a-b)c. \qquad$ Art. 48

Dividing by abc, $\qquad \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}.$

Therefore $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P. The process reversed proves the converse.

Cor. 1.—The general expression for an harmonical series is obtained by taking the reciprocals of the successive terms of an arithmetical series.

Thus $$\frac{1}{a}, \ \frac{1}{a+d}, \ \frac{1}{a+2d} \cdots \frac{1}{a+(n-1)d}$$

is a general expression for any harmonical series.

Cor. 2.—A constant quantity divided by the successive terms of an A. P. gives quotients in H. P. For the reciprocal of $\dfrac{c}{\{a+(n-1)d\}}$ is $\dfrac{a}{c}+(n-1)\dfrac{d}{c}$, the general term of an A. P. whose first term is $\dfrac{a}{c}$ and common difference $\dfrac{d}{c}$.

115. Harmonical progression, though connected with arithmetical progression by the simple relation given in the former Art., is, nevertheless, essentially different, in several respects, from both it and geometrical progression.

The various terms in A. P. and in G. P. can be written when the first term and the common difference, or the common ratio, are given; but in H. P. there is no quantity corresponding to the common difference or the common ratio. Again, in the other progressions convenient expressions can be found which represent the sum of any number of terms; but no corresponding expression can be found for the sum of an harmonical series. Problems in H. P. are always solved by using the corresponding arithmetical progression.

116. If two terms of an harmonical progression be given, the series is completely determined.

For let the m^{th} and n^{th} terms be p and q.

Then $$a+(m-1)d=\frac{1}{p}, \qquad \text{Art. 114}$$

and $$a+(n-1)d=\frac{1}{q}.$$

From these equations a and d may be found. These determine the A. P. the reciprocals of whose terms are the H. P. required.

117. *To find the harmonic mean between two given extremes.*

Let a and b denote the given extremes and x the required mean, so that a, x, b are in H. P.

Then $$a : b = a - x : x - b.$$

Therefore $$a(x-b) = b(a-x),$$

from which $$x = \frac{2ab}{a+b}.$$

Thus the harmonic mean between two quantities is *twice their product divided by their sum.*

118. *To insert a given number of harmonic means between two given extremes.*

Let a and b be the given extremes and m the number of means. Insert m arithmetic means between $\frac{1}{a}$ and $\frac{1}{b}$; their reciprocals will be the harmonic means required.

The arithmetic means are:

$$\frac{1}{a} + \frac{1}{m+1}\left(\frac{1}{b} - \frac{1}{a}\right), \quad \frac{1}{a} + \frac{2}{m+1}\left(\frac{1}{b} - \frac{1}{a}\right) \dots \frac{1}{a} + \frac{m}{m+1}\left(\frac{1}{b} - \frac{1}{a}\right).$$

Simplifying, and taking the reciprocals, we get

$$\frac{(m+1)ab}{mb+a}, \quad \frac{(m+1)ab}{(m-1)b+2a} \dots \frac{(m+1)ab}{b+ma},$$

the harmonic means required.

This series can easily be remembered by observing that the numerators are all alike, and that the denominators are the same as the numerators of the corresponding arithmetic means taken in the reverse order.

Cor.—The product of the r^{th} arithmetic and the $(m-r+1)^{\text{th}}$ harmonic means between any two quantities is equal to the square of the geometric mean between the same two quantities.

119. *If A, G and H denote the arithmetic, the geometric and the harmonic means between any two quantities, then A, G and H are in geometrical progression, and A is the greatest or the least according as the quantities are both positive or both negative.*

Let a and b denote the two quantities.

Then $$A=\frac{(a+b)}{2},\quad G^2=ab,\quad H=\frac{2ab}{a+b},$$

and $$AH=\frac{a+b}{2}\cdot\frac{2ab}{a+b}$$
$$=ab$$
$$=G^2.$$

Therefore G is the geometric mean between A and H.

Again, $$A-G=\frac{1}{2}(a+b)-\sqrt{ab}$$
$$=\frac{1}{2}(a-2\sqrt{ab}+b)$$
$$=\frac{1}{2}(\sqrt{a}-\sqrt{b})^2,$$

a square quantity which is positive if a and b are positive, therefore A is greater than G; and A, G, H are in geometrical progression, therefore G is greater than H.

If a and b are negative, $\sqrt{a}$ and $\sqrt{b}$ are *imaginary*, and the square quantity is *negative*: therefore A is less than G, and G is less than H.

If one quantity is positive and the other negative, A and H are real, but G is imaginary, and consequently no comparison in magnitude can be made between it and the other two.

120. *Ex. 1.*—The arithmetic mean between two numbers exceeds the geometric by 55, and the geometric exceeds the harmonic by 44; find the numbers.

Let x denote the arithmetic mean.

Then $x-55$ and $x-99$ will denote the geometric and harmonic means; and since A, G, H are in geometrical progression,

therefore $$x(x-99)=(x-55)^2,$$

from which $$x=275.$$

The three means, then, are 275, 220 and 176.

Next let x and y denote the required numbers.

Then $$x+y=550, \qquad (1)$$

and $$\sqrt{xy}=220. \qquad (2)$$

Adding twice (2) to (1), and taking the square root, we get

$$\sqrt{x}+\sqrt{y}=3\sqrt{110}.$$

Similarly $$\sqrt{x}-\sqrt{y}=\sqrt{110}.$$

Therefore $$\sqrt{x}=2\sqrt{110} \text{ and } \sqrt{y}=\sqrt{110},$$

or $$x=440 \text{ and } y=110,$$

the numbers required.

Ex. 2.—If $\dfrac{a}{b+c}$, $\dfrac{b}{c+a}$, $\dfrac{c}{a+b}$ are in H. P., then a, b, c are also in H. P., and conversely.

Since $$\frac{a}{b+c},\ \frac{b}{c+a},\ \frac{c}{a+b} \text{ are in H. P.,}$$

therefore $$\frac{b+c}{a},\ \frac{c+a}{b},\ \frac{a+b}{c} \text{ are in A. P.}$$

Add a unit to each,

then $$\frac{b+c+a}{a},\ \frac{c+a+b}{b},\ \frac{a+b+c}{c} \text{ are in A. P.}$$

Divide by $a+b+c$,

then $$\frac{1}{a},\ \frac{1}{b},\ \frac{1}{c} \text{ are in A. P.}$$

Therefore a, b, c are in H. P.

The work reversed proves the converse.

Ex. 3.—If $\dfrac{px}{a-x}=\dfrac{qy}{a-y}=\dfrac{rz}{a-z}$, and if p, q, r are in A. P., then x, y, z are in H. P.

Put $$\frac{px}{a-x}=\frac{qy}{a-y}=\frac{rz}{a-z}=k,$$

then $$\frac{x}{a-x}=\frac{k}{p},\quad \frac{y}{a-y}=\frac{k}{q},\quad \frac{z}{a-z}=\frac{k}{r}.$$

Therefore $$\frac{x}{a-x},\ \frac{y}{a-y},\ \frac{z}{a-z} \text{ are in H. P.,}$$ {Art. 114, Cor. 2.

and $$\frac{a-x}{x},\ \frac{a-y}{y},\ \frac{a-z}{z} \text{ are in A. P.}$$

Add a unit to each,

then $$\frac{a}{x},\ \frac{a}{y},\ \frac{a}{z} \text{ are in A. P.}$$

Therefore $$x,\ y,\ z \text{ are in H. P.}$$

EXERCISE X.

1. Find the tenth term of the series, 3, 4, 6 ....

2. Find the n^{th} term of the series, $\frac{2}{3}$, $\frac{4}{5}$, 1 ....

3. Find the twenty-fourth term of the series, 24, 12, 8 ....

4. Find the arithmetic, the geometric and the harmonic means between 2 and 32.

5. Insert two harmonic means between 1 and 2.

6. Insert three harmonic means between 16 and 4.

7. The second and fourth terms of an H. P. are $\frac{1}{2}$ and $-\frac{1}{2}$; find the first, third, fifth and n^{th} terms.

8. The first and second terms of an H. P. are a and b; find the n^{th} term.

9. The arithmetic and geometric means between two quantities are $a-b$ and $\sqrt{a^2-b^2}$ respectively; find the harmonic mean.

10. The arithmetic and harmonic means between two numbers are 2 and $1\frac{1}{2}$ respectively; find the numbers.

11. The sum and difference of the arithmetic and geometric means between two numbers are 16 and 4 respectively; find the harmonic mean.

12. Find a number such that the arithmetic mean between it and 2 may be $2\frac{2}{7}$ times the harmonic mean.

13. Find two numbers whose difference is $16\frac{1}{4}$, and the geometric mean between the arithmetic and harmonic means of which is 9.

14. The sum of three terms of an H. P. is $\frac{11}{12}$, and the first term is $\frac{1}{2}$; find the series and continue it two terms each way.

15. The arithmetic mean between two numbers exceeds the geometric mean by 13, and the geometric mean exceeds the harmonic mean by 12; find the numbers.

16. From each of three quantities in H. P. what quantity must be taken away so that the remainders may be in G. P.?

17. The sum of three numbers in H. P. is 11, and the sum of their squares is 49; find the numbers.

18. Find the value of $\dfrac{a(b-c)}{a-b}$ when a, b, c are (1) in **A. P.**, (2) in G. P., and (3) in H. P.

19. If a, b, c are in A. P., and a, mb, c in G. P., prove that a, m^2b, c are in H. P.

20. If x is the harmonic mean between m and n, show that

$$\frac{1}{x-m}+\frac{1}{x-n}=\frac{1}{m}+\frac{1}{n}.$$

21. If four quantities are proportionals, and the first three are in A. P., prove that the last three are in H. P.

22. If H be the harmonic mean between a and b, prove that it is also the harmonic mean between $H-a$ and $H-b$.

23. If a, b, c be in H. P.,

then $$\frac{b+a}{b-a}+\frac{b+c}{b-c}=2,$$

and $$b^2(a-c)^2=2\{c^2(b-a)^2+a^2(c-b)^2\}.$$

24. If between each two successive terms of the series, a, ar, $ar^2 \ldots ar^n$, an harmonic mean be inserted, the sum of those means will be $$\frac{2ar(r^n-1)}{r^2-1}.$$

25. If a, b, c be in H. P., then $\frac{bc}{b+c}$, $\frac{ca}{c+a}$, $\frac{ab}{a+b}$ are also in H. P., and $\frac{b+c}{a}$, $\frac{c+a}{b}$, $\frac{a+b}{c}$ are in A. P.

26. If a, $2b$ and c be in H. P., then will $a+c$, a and $a-b$ be in G. P.; so also will $c+a$, c and $c-b$ be in G. P.

27. Find the condition that a, b, c may be the p^{th}, q^{th} and r^{th} terms of an harmonic progression.

28. If the $(p+q)^{\text{th}}$ term of an harmonic progression be m and the $(p-q)^{\text{th}}$ term be n, find the p^{th} and q^{th} terms.

29. If the p^{th} term of an harmonic progression be P, and the q^{th} term be Q, the n^{th} term will be $$\frac{(p-q)PQ}{(n-q)Q-(n-p)P}.$$

30. If a, b, c be in A. P., and a, b, d in H. P., then $$\frac{c}{d}=1-\frac{2(a-b)^2}{ab}.$$

31. If A, G, H be the arithmetic, geometric and harmonic means between a and b, then $$\frac{H}{A}=1+\frac{(H-a)(H-b)}{G^2} \text{ and } \frac{1}{A+G}+\frac{1}{G+H}=\frac{1}{G}.$$

32. If $ax=by=cz$, and a, b, c are in A. P., then x, y, z are in H. P.

33. If $a^x = b^y = c^z$, and a, b, c are in G. P., then x, y, z are in H. P.

34. If either of two consecutive terms of an H. P. is divisible by their difference, show that one of the terms of the series is infinity. Can any term of a finite harmonical series be zero?

35. If an arithmetical and an harmonical progression have each the same first and second terms, a and b, and if x and y be the n^{th} terms in the two series, then

$$\frac{y(x-a)}{a(y-b)} = \frac{n-1}{n-2}.$$

36. If between any two quantities there are inserted two arithmetic means, A_1, A_2, two geometric means, G_1, G_2, two harmonic means, H_1, H_2, then

$$\frac{A_1 + A_2}{H_1 + H_2} = \frac{G_1 G_2}{H_1 H_2} \text{ and } A_1 H_2 + A_2 H_1 = 2G_1 G_2.$$

37. If a, b, c are in H. P., then $\dfrac{a}{b+c-2a}$, $\dfrac{b}{c+a-2b}$, $\dfrac{c}{a+b-2c}$ are also in H. P.

38. If a, b, c are in H. P., then $\dfrac{1}{a} + \dfrac{1}{b+c}$, $\dfrac{1}{b} + \dfrac{1}{a+c}$, $\dfrac{1}{c} + \dfrac{1}{a+b}$ are in H. P.; and if $a + \dfrac{bc}{b+c}$, $b + \dfrac{ca}{c+a}$, $c + \dfrac{ab}{a+b}$ are in H. P., then a, b, c are in A. P.

39. If a, b, c be three quantities such that a is the arithmetic mean between b and c, and c is the harmonic mean between a and b, then a, b, c are in G. P.

40. If the harmonic means between each pair of the three quantities, a, b, c be in A. P., then b^2, a^2, c^2 shall be in H. P.; but if the harmonic means be in H. P., then b, a, c shall be in H. P.

41. If S_1, S_2, S_3 denote the sums of n terms of each of three arithmetical series having the same first term, a, and their common differences in H. P., then

$$an = \frac{2S_1 S_3 - S_2(S_1 + S_3)}{S_1 + S_3 - 2S_2}.$$

CHAPTER VIII.

SQUARES AND CUBES OF THE NATURAL NUMBERS.

121. In the practical application of mathematics series frequently present themselves whose summation depends upon the sum of the squares or the cubes of the natural numbers. We therefore give their summation, with a few simple applications.

122. *To find the sum of the squares of the first n natural numbers.*

Denote the sum by S.

Then $$S = 1^2 + 2^2 + 3^2 + \dots n^2.$$

We have identically,

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1,$$
$$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1,$$
$$(n-2)^3 - (n-3)^3 = 3(n-2)^2 - 3(n-2) + 1,$$
$$\dots\dots\dots\dots$$
$$3^3 - 2^3 = 3 \cdot 3^2 - 3 \cdot 3 + 1,$$
$$2^3 - 1^3 = 3 \cdot 2^2 - 3 \cdot 2 + 1,$$
$$1^3 - 0^3 = 3 \cdot 1^2 - 3 \cdot 1 + 1.$$

Then, by addition,

$$n^3 = 3(1^2 + 2^2 + \dots n^2) - 3(1 + 2 + \dots n) + n$$
$$= 3S - \frac{3n(n+1)}{2} + n.$$

Therefore $$3S = n^3 - n + \frac{3n(n+1)}{2}$$

or $$S = \frac{n(n+1)(2n+1)}{6}.$$

It should be observed that the preceding equalities are *identities*, and consequently true *for all values of* n. The first is obtained by subtraction; the second is obtained from the first by changing n into $n-1$, and so on; the final equalities are obtained by substituting numbers for n. Upon adding, the second column on the left cancels the first, excepting the single term, n^3.

Ex.—Sum to n terms the series,

$$1.2+2.3+3.4+\ldots\ldots n(n+1).$$

The series may be written

$$\begin{aligned} S &= (1^2+1)+(2^2+2)+(3^2+3)+\ldots\ldots(n^2+n) \\ &= (1^2+2^2+3^2+\ldots\ldots n^2)+(1+2+3+\ldots\ldots n) \\ &= \frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2} \\ &= \frac{n(n+1)(n+2)}{3}. \end{aligned}$$

123. *To find the sum of the cubes of the first n natural numbers.*

Denote the sum by S.

Then $$S=1^3+2^3+3^3+\ldots\ldots n^3.$$

We have identically,

$$\begin{aligned} n^4-(n-1)^4 &= 4n^3-6n^2+4n-1, \\ (n-1)^4-(n-2)^4 &= 4(n-1)^3-6(n-1)^2+4(n-1)-1, \\ &\ldots\ldots\ldots\ldots \\ 2^4-1^4 &= 4.2^3-6.2^2+4.2-1, \\ 1^4-0^4 &= 4.1^3-6.1^2+4.1-1. \end{aligned}$$

Then, by addition,

$$n^4=4S-6(1^2+2^2+\ldots\ldots n^2)+4(1+2+\ldots\ldots n)-n.$$

Therefore $$\begin{aligned} 4S &= n^4+n+n(n+1)(2n+1)-2n(n+1) \\ &= n(n+1)(n^2-n+1+2n+1-2) \\ &= n(n+1)(n^2+n), \end{aligned}$$

or $$S=\left\{\frac{n(n+1)}{2}\right\}^2.$$

Thus the sum of the cubes of the natural numbers is equal to the square of the sum of the numbers.

The same method may be applied to find the sum of the fourth, fifth, etc., powers in succession.

124. The sum of the cubes of the natural numbers may easily be found independently of the sum of their squares as follows:

Arrange the odd numbers in groups thus:

$$1+(3+5)+(7+9+11)+(13+15+17+19)+\ldots.$$

In n groups there are $1+2+3\ldots.n=\dfrac{n(n+1)}{2}$ terms. The last term of the n^{th} group is n^2+n-1, the first term is n^2-n+1, and there are n terms in it; therefore the sum of the n^{th} group is

$$\{(n^2-n+1)+(n^2+n-1)\}\frac{n}{2}=n^3.$$

This shows that the successive groups are the cubes of the natural numbers. To find their sum remove the brackets, thus giving $\dfrac{n(n+1)}{2}$ terms of the series of odd numbers, whose sum is at once known to be $\left\{\dfrac{n(n+1)}{2}\right\}^2$.

125. It is frequently convenient to indicate by a single symbol that the whole of a series of terms is to be taken. This is generally done by writing the Greek letter Σ before the n^{th} term of the series, thus:

$1+2+3+\ldots.n$	is denoted by Σn.
$1^2+2^2+3^2+\ldots.n^2$.	" " " Σn^2.
$a+ar+ar^2\ldots.ar^{n-1}$	" " " Σar^{n-1}.

When more than one letter follows the sign of summation, care must be taken to correctly distinguish the *variable* and *constants*. In the latter example n is the variable; if r were taken for the variable, Σar^{n-1} would stand for $a+2^{n-1}a+\ldots.r^{n-1}a$. The context will usually decide the point; in doubtful cases the variable must be specified.

Ex.—Sum to n terms the series,

$$1.2.3+2.3.4+3.4.5+\dots n(n+1)(n+2).$$

The n^{th} term may be written n^3+3n^2+2n.

$$\therefore \text{ sum of } n \text{ terms} = \Sigma n^3 + 3\Sigma n^2 + 2\Sigma n$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}.$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}.$$

126. When the terms of a series are alternately positive and negative it is sometimes necessary to consider separately the cases in which n is even or odd. The two results can then be combined as follows:

Let A and B denote any two quantities.

Then $A+(-1)^n B$ denotes their *sum* when n is *even*, but their *difference* when n is *odd*.

Let p and q denote the sums of the series when n is even and odd respectively.

Then let $\qquad A+B=p$

and $\qquad A-B=q,$

from which $\qquad A=\frac{1}{2}(p+q),\ B=\frac{1}{2}(p-q).$

Therefore $\frac{1}{2}\{(p+q)+(-1)^n(p-q)\}$ is the required sum whether n be even or odd.

Ex.—Sum $1-2+3-4+5-\dots$ to n terms.

When n is even,

$$S=(1-2)+(3-4)\dots \text{ to } \frac{n}{2} \text{ groups}$$

$$=-1-1-\dots \text{ to } \frac{n}{2} \text{ terms}$$

$$=-\frac{n}{2}, \text{ the sum when } n \text{ is } \textit{even}.$$

When n is odd,

$$S = 1 - (2-3) - (3-4) - \ldots \text{ to } \frac{n-1}{2} \text{ groups.}$$

$$= 1 + 1 + 1 + \ldots \text{ to } 1 + \frac{n-1}{2} \text{ terms}$$

$$= \frac{n+1}{2}, \text{ the sum when } n \text{ is } \textit{odd}.$$

Then $$\frac{1}{2}(p+q) = \frac{1}{4}, \quad \frac{1}{2}(p-q) = -\frac{2n+1}{4}$$

and $$S = \frac{1}{4} - (-1)^n \frac{2n+1}{4}$$

$$= \frac{1}{4}\{1 + (-1)^{n+1}(2n+1)\},$$

whether n be even or odd.

PILES OF SHOT AND SHELL.

127. *To find the number of shot arranged in a complete pyramid on a square base.*

Let n be the number of shot on a side of the lowest layer; then $n-1$, $n-2$, etc., will be the numbers on a side of the successive higher layers. The number of shot composing the layers will be n^2, $(n-1)^2$, etc., ending with a single shot at the top of the pile.

Then $$S = 1^2 + 2^2 + \ldots (n-1)^2 + n^2$$

$$= \frac{n(n+1)(2n+1)}{6}. \qquad \text{Art. 122}$$

128. *To find the number of shot arranged in a complete pyramid whose base is an equilateral triangle.*

Let n be the number of shot on a side of the base. Counting

the shot in the lowest, or n^{th}, layer by rows we see that it contains

$$n+(n-1)+(n-2)\ldots\ldots+1=\frac{n(n+1)}{2}.$$

Similarly the $(n-1)^{\text{th}}$ layer contains $\dfrac{(n-1)(n)}{2}$, etc.

We have thus to find the sum of n terms of the series whose n^{th} term is $\dfrac{1}{2}(n^2+n)$.

Therefore $$S=\frac{1}{2}(\Sigma n^2+\Sigma n)$$

$$=\frac{n(n+1)(n+2)}{6}. \qquad \text{Art. 122, } Ex.$$

The number of shot in the successive layers, beginning at the top, are 1, 3, 6, 10, 15, etc., which are called *triangular* numbers for the same reason that 1, 4, 9, 16, etc., are called *square* numbers.

129. *To find the number of shot contained in a complete pile upon a rectangular base.*

Let m and n be the number of shot in a side and an end of base.

There will be n layers, the top one consisting of a single row containing $m-n+1$ shot. Each succeeding layer will consist of one more row, and each row will contain one more shot than the preceding. The numbers in the successive layers will be the terms of the following series.

$$\begin{aligned}\therefore\ S&=(m-n+1)+2(m-n+2)+3(m-n+3)\ldots\ldots n(m-n+n)\\&=(m-n)(1+2+3+\ldots\ldots n)+(1^2+2^2+3^2+\ldots\ldots n^2)\\&=\frac{(m-n)n(n+1)}{2}+\frac{n(n+1)(2n+1)}{6}\\&=\frac{n(n+1)(3m-n+1)}{6}.\end{aligned}$$

If $m=n$ the rectangle becomes a square, and this result reduces to that obtained for the number on a square base.

130. *To find the number of shot in an incomplete pile.*

Find by the preceding Arts. the number the pile would contain if complete, and from the result subtract the number required to complete it.

Ex.—Find the number of shot in an incomplete pile of six layers, there being 20 shot in a side and 12 in the end of the base.

If complete the pile would contain $\dfrac{12 \,.\, 13 \,.\, 49}{6} = 1274$ shot.

The number required to complete it is $\dfrac{6 \,.\, 7 \,.\, 37}{6} = 259$ shot.

Therefore the number of shot in the pile is $1274 - 259 = 1015$.

131. Given the number of shot in a complete square, or in a complete triangular, pile, to find the number in one side of the base.

Let N be the number of shot in the pile and n the number in a side of the base. Then

(1) In the triangular pile $n(n+1)(n+2) = 6N$. Now, $n(n+1)(n+2) > n^3$ but $< (n+1)^3$, therefore n is the integral part of the cube root of $6N$.

(2) In the square pile $n(n+1)\left(n+\dfrac{1}{2}\right) = 3N$, therefore, as before, n is the integral part of the cube root of $3N$.

EXERCISE XI.

1. Find the number of shot in a complete pile on a square base containing 20 shot on a side.

2. Find the number of shot in a complete pile on a triangular base containing 30 shot on a side.

3. Find the number of shot in a complete pile on a rectangular base containing 25 and 20 shot on the side and the end respec tively.

4. How many shot in an incomplete pile of twelve courses on a triangular base containing 37 shot on a side?

5. How many shot in an incomplete pile of eleven courses on a rectangular base containing 27 and 23 shot on the side and the end respectively?

6. An incomplete square pile contains 225 shot in the top layer and 729 in the bottom; how many shot in the pile?

7. The base of an incomplete rectangular pile contains 800 shot and the top 450; the length of the base is greater than the breadth by 7 shot; how many shot in the whole pile?

8. A triangular and a square pile of shot have each the same number on a side of the base, but the former contains only four-sevenths as many shot as the latter; find the number in each pile.

9. How many shot in an incomplete triangular pile of eleven courses, there being 155 more shot in the bottom layer than in the top?

10. Show that the number of shot in a square pile is one-fourth the number in a triangular pile of double the number of courses.

11. If from a complete square pile of shot a triangular pile of the same number of courses be formed, show that the remainder will just form another triangular pile.

12. The number of shot in an incomplete square pile is equal to six times the number required to complete it; and the number of completed courses is equal to the number of courses required to complete the pile; find the number of shot in the incomplete pile.

13. Find the number of shot in the rectangular pile in which the number in the lowest course is 600, and in the top ridge, 11.

14. How many courses must be taken from the top of a complete square pile of shot to make up 2,870? How many more courses will make 12,040?

15. The value of a triangular pile of 16-lb. shot is $244.80; if

the value of the iron be $2.25 per cwt., find the number of shot in the lowest layer.

16. A triangular and a square pile of shot have each the same number in a side of the base, and also one base row in common; the number of shot in both piles is 64,975; how many in a row in the base, both piles being complete?

17. How many 8-inch shells would there be in a square pile erected on the ground formerly covered by a square pile containing 4,900 13-inch shells, both piles being complete?

18. The numbers of shot in a triangular, a square and a rectangular pile are in A. P., the number of courses, n, being the same in each; show that $n-1$ is a multiple of 3. If the rectangular pile contains 5,566 shot, how many does each of the others contain?

19. Four equal square piles of shot are so placed that their bases form a larger square, and each pile has two base rows in common with the adjacent piles. There are 21 shot in the base row of the large square; how many shot in the whole? If there are 3,225 shot in all, how many in a base row of the large square?

EXERCISE XII.

MISCELLANEOUS EXAMPLES IN PROGRESSION.

Sum to n terms the series:

1. $2^2+4^2+6^2 \dots$
2. $1^2+3^2+5^2 \dots$
3. $a^2+(a+1)^2+(a+2)^2 \dots$
4. $1^3+3^3+5^3 \dots$
5. $1^2.2+2^2.3+3^2.4 \dots$
6. $n+2(n-1)+3(n-2) \dots$
7. $1^2-2^2+3^2- \dots$
8. $1^2-3^2+5^2- \dots$

9. Find the sum of the squares of n terms of the series whose n^{th} term is $2-3n$.

10. If S_n denotes the sum of the first n natural numbers, find the sums of $S_1+S_2+\dots S_n$ and of $S_p+S_{2p}+\dots S_{np}$.

11. Sum $\frac{n^3+1^3}{n+1}+\frac{n^3+2^3}{n+2}+\frac{n^3+3^3}{n+3}\ldots$ to n terms.

12. Given that $x+2\left\{x-\frac{1}{n-1}\right\}+3\left\{x-\frac{2}{n-1}\right\}\ldots$ to n terms is zero, find x.

13. Given x, y, z in G. P., y, z, 4 in H. P., $\frac{1}{4}$, x, y in A. P., find x, y and z.

14. Sum to infinity the series:

(1) $\frac{1^2}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\frac{4^2}{2^4}\ldots$ (2) $\frac{1^2}{5}-\frac{2^2}{5^2}+\frac{3^2}{5^3}-\frac{4^2}{5^4}\ldots$

15. Divide the number $2^{2n}-1$ into n parts in the ratio of 1, 2, 4, 8, etc.

16. Find the sum of the products of the first n natural numbers taken two and two together.

17. Show that $\frac{c+2a}{c-b}+\frac{c+2b}{c-a}$ is equal to 4, greater than 7 or 10, according as a, c, b are in A., G. or H. P.

18. The three sides of a triangle and the perpendicular from the opposite angle on the greatest side are in G. P.; show that the triangle is right-angled.

19. If 1, x, x^3 and 1, y^2, y^3 be each in H. P., show that $-y^2$, y, x, x^2 are in A. P., and that their sum is x^3+y^3, supposing $x+y$ not to be zero and x and y not to be unity.

20. Sum $1.1.3+2.3.5+3.5.7+4.7.9\ldots$ to n terms.

21. Let the sums of the squares of the roots of the equations,

$$\frac{1}{3}\{x^2+(m+1)x\}=1,\quad \frac{2}{3}\{x^2+(m+2)x\}=1,\quad \frac{3}{3}\{x^2+(m+3)x\}=1,$$

be A, B and C; find the value of m so that A, B and C may be in G. P.

22. Between the successive terms of an A. P. arithmetical means are inserted, one between the first and second, two be-

tween the second and third, three between the third and fourth, and so on; find the sum of all the means so inserted, the first series consisting of $n+1$ terms.

23. Sum $1^3 - 2^3 + 3^3 - 4^3 + \ldots$ to n terms.

24. If A_1 be the arithmetic mean between a and b, A_2 the arithmetic mean between A_1 and b, and so on, show that

$$A_1 + A_2 + \ldots A_n = a + (n-1)b - \frac{a-b}{2^n}.$$

25. Sum $2.5 - 3.9 + 4.13 - 5.17 + \ldots$ to $2n$ terms and to n terms.

26. If $a, a_1, a_2, a_3 \ldots a_n$ be in A. P., and $a, b_1, b_2 \ldots b_n$ be in G. P., and if r, the common ratio of the latter series, be equal to the common difference of the former series, then

$$(a_1 ar - ab_1) + (a_2 b_1 r - a_1 b_2) + \ldots (a_n b_{n-1} r - a_{n-1} b_n) = \frac{ar^2(r^n - 1)}{r-1}.$$

27. Show that the sum to n terms of the series whose $(m-1)^{\text{th}}$ term is $m(m-1)$ is equal to $\frac{1}{3}$ of the product of the n^{th} terms of the three series whose $(p-1)^{\text{th}}$ terms are $p-1$, p and $p+1$.

28. An A. P., a G. P. and an H. P. have each the same first and last terms and the same number of terms, n; and their r^{th} terms are a_r, b_r, c_r; prove $a_{r+1} : b_{r+1} = b_{n-r} : c_{n-r}$, and if A, B, C be the continued products of the n terms, then $AC = B^2$.

29. Between two quantities an harmonic mean is inserted, and between each pair a geometric mean is inserted; the three means are in A. P.; prove that the ratio of the two quantities is

$$7 \pm 4\sqrt{3} : 1.$$

30. If a, b, c, d are in G. P.,

then $$abcd\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)^2 = (a+b+c+d)^2,$$

and $$\frac{\sqrt{a^4+b^4} + \sqrt{b^4+c^4} + \sqrt{c^4+d^4}}{(a^2+b^2)^{-1} + (b^2+c^2)^{-1} + (c^2+d^2)^{-1}} = \frac{b^4}{a^4}(a^2+b^2)\sqrt{a^4+b^4}.$$

31. If a, b, c are in G. P., and p, q the arithmetic means between a, b and b, c, then b will be the harmonic mean between p and q.

32. If a, b, c are in A. P., α, β, γ in H. P., and $\frac{a}{\gamma}+\frac{\gamma}{a}=\frac{a}{c}+\frac{c}{a}$, then $a\alpha$, $b\beta$, $c\gamma$ are in G. P.

33. Three numbers are in G. P., but if each be increased by 15 they will be in H. P. The sum of the numbers is 49; find them.

34. Prove that $a_2a_3-a_1a_4$ is positive, zero or negative according as a_1, a_2, a_3, a_4 are in A. P., G. P. or H. P.

35. If m and n be any two numbers, g their geometric mean, a_1, h_1 and a_2, h_2 the arithmetic and harmonic means between m, g and g, n respectively, then $a_1h_2=g^2=a_2h_1$.

36. If a, b, c be in A. P., α, β, γ in H. P., and $a\alpha$, $b\beta$, $c\gamma$ in G. P., then

$$a:b:c=\frac{1}{\gamma}:\frac{1}{\beta}:\frac{1}{\alpha}.$$

37. If n be the first of two arithmetic means between two numbers, and m the first of two harmonic means between the same two numbers, prove that the value of m does not lie between the values of n and $9n$.

38. If x, y, z be in A. P., ax, by, cz in G. P., and a, b, c in H. P., then the ratio of the harmonic means between a and c, x and z is equal to the ratio of the geometric means of the same quantities.

39. If A_1, $A_2 \ldots A_n$ be the n arithmetic means, and H_1, $H_2 \ldots H_n$ the n harmonic means, between a and b, find the sum to n terms of the series whose r^{th} term is

$$\frac{(A_r-a)(H_r-a)}{H_r}.$$

40. If a_r, b_r be the arithmetic and geometric means between a_{r-1} and b_{r-1}, show that

$$a_{n-2}=\left\{a_n^{\frac{1}{2}}\pm(a_n^2-b_n^2)^{\frac{1}{4}}\right\}^2,$$

$$b_{n-2}=\left\{a_n^{\frac{1}{2}}\mp(a_n^2-b_n^2)^{\frac{1}{4}}\right\}^2.$$

41. If a, b, c are in A. P., G. P. or H. P., then $a^n+c^n>2b^n$.

42. If a_1, $a_2 \ldots . a_n$ are in H. P., then

$$\frac{a_1}{ma_1+a_2+\ldots . a_n},\quad \frac{a_2}{a_1+ma_2+\ldots . a_n}\ \ldots .\ \frac{a_n}{a_1+a_2+\ldots . ma_n}$$

are also in H. P.

43. If the squared differences of p, q, r be in A. P., then the differences in order are in H. P.

44. If P, Q, R be the p^{th}, q^{th} and r^{th} terms of an H. P., then

$$\left\{\frac{q^2-r^2}{P^2}+\frac{r^2-p^2}{Q^2}+\frac{p^2-q^2}{R^2}\right\}^2=4\left\{\frac{q-r}{P^2}+\frac{r-p}{Q^2}+\frac{p-q}{R^2}\right\}\left\{\frac{qr(q-r)}{P^2}+\ldots\right\}.$$

45. If n arithmetical and n harmonical means be inserted between two quantities, a and b, and a series of n terms formed by dividing each arithmetical by the corresponding harmonical mean, the sum of the series will be

$$n\left\{1+\frac{(n+2)(a-b)^2}{6(n+1)ab}\right\}.$$

46. Find the sum of n terms of the series,

$$1+2^2+3+4^2+5+6^2\ldots .$$

(1) when n is even, (2) when n is odd; find the n^{th} term.

47. Find the n^{th} term and the sum of n terms of the series,

$$2+2(2^2+2)+6+2(4^2+4)+10+2(6^2+6)+14+2(8^2+8)\ldots .$$

(1) when n is even, (2) when n is odd, (3) when n is any positive integer.

48. Between each of the pairs of quantities, (x, y), $(x, 2y)$, $(x, 4y)$, etc., are inserted m geometric means, and M_r is the m^{th} mean of the r^{th} pair; show that $\dfrac{M_{r+1}}{M_r}=2^{\frac{m}{m+1}}$ for all values of r

49. There are n piles of stones placed in a straight line, the intervals between them being 3, 5, 7 .... $2n-1$ yards, and the piles containing 1, 2, 3 .... n stones respectively. How far must a person walk to gather them singly into a heap one yard beyond the end of the row at which is placed the single stone?

CHAPTER IX.

SCALES OF NOTATION.

132. The basis of number is the unit or elementary number *one;* all other numbers are repetitions of this unit. The groups of units thus formed by successively adding one are known by distinctive names, and are represented by symbols. Thus "two" is the name given to "one and one"; "three" is the name given to "two and one," and so on. These groups are represented by the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, called digits, each of which represents a unit more than the preceding. We have no symbol to represent "9 and 1"; we therefore give it a distinct name (ten), and represent it as a unit of a second order. To distinguish it from the simple unit we write the symbol 0 beside it on the right. When other units are added they take the place of the 0 until two units of the second order are reached. Ten units of the second order are expressed as a unit of the third order (named hundred), and so on, the order of any unit being determined by the position of the digit representing it, counting from the right.

This method of representing numbers is called the **Common, Denary** or **Decimal Scale of Notation,** and ten is said to be the **radix,** or base, of the system.

133. In like manner numbers may be represented by assuming a different radix and using the figures 1, 2, 3, etc., in the same sense as before. The number of independent significant symbols must evidently be one less than the radix. Thus if *eight* be taken for the radix, the figures 8 and 9 will be unnecessary; but if *twelve* be taken, two new symbols must be introduced to represent "ten" and "eleven" respectively. The letters t and e are generally used for this purpose.

The exact meaning of each symbol employed should be carefully observed. Thus in the common scale

$$365 \text{ means } 3 \text{ times } 10^2 + 6 \text{ times } 10 + 5,$$

but in the scale with radix 8

$$365 \text{ means } 3 \text{ times } 8^2 + 6 \text{ times } 8 + 5;$$

and generally, if a_0, a_1, $a_2 \ldots . a_n$ denote the digits in order, beginning with the units, r the radix, then the number is

$$a_n r^n + a_{n-1} r^{n-1} + a_{n-2} r^{n-2} + \ldots . a_2 r^2 + a_1 r^1 + a_0$$

where a_n, $a_{n-1} \ldots . a_0$ are all positive integers, each less than r, but any one after the first may be zero.

The radix itself is always represented by 10.

134. Our language, being adapted to the decimal notation, is inapplicable to any other. Thus 26 in the scale of *eight* must not be read *twenty-six*, for it is not *two tens* and six, but *two eights* and six, or twenty-two. Since we have no words to designate the numbers in the form in which they appear in the various scales, we read such numbers by naming the digits in order, giving the radix of the scale.

135. The various arithmetical operations can be performed in any scale on the principles which are employed in the common scale. But inasmuch as we are not familiar with the addition and multiplication tables in any but the common scale, we shall be compelled to determine the *carrying figures* by an indirect process, as shown in the following example:

Ex. 1.—Multiply 2763 by 25 in the nonary scale.

$$\begin{array}{r} 2763 \\ 25 \\ \hline 15246 \\ 5636 \\ \hline 72616 \end{array}$$

In this scale we carry one for every nine. Multiplying by 5 we have

$$3 \times 5 = \text{fifteen} = 9 + 6 = 16.$$

We therefore set down 6 and carry 1; then

$$6 \times 5 + 1 = \text{thirty-one} = 3 \text{ times } 9 + 4 = 34,$$

and so on till the multiplication is complete.

The same method is followed in the addition.

Ex. 2.—Divide $59t46e3$ by 7 in the duodenary scale.

$$\begin{array}{r|l} 7 & \underline{59t46e3} \\ & 9e9285 \text{ rem. } 4 \end{array}$$

The first two figures, 59, are not fifty-nine, but 5 times twelve $+ 9 =$ sixty-nine. Dividing by 7 we get 9 for quotient and 6 remainder. Next, 6 times twelve $+ t =$ eighty-two; dividing by 7 we get eleven (or e) for quotient and 5 remainder, and so on.

136. *To express a given number in a scale with a given radix.*

Let N denote the number, r the radix. Divide N by r, the quotient by r, the second quotient by r, and so on until the last quotient is less than r. Denote the successive quotients by $Q_1, Q_2 \ldots . Q_{n-1}, p_n$, and the remainders by $p_0, p_1, p_2 \ldots . p_{n-1}$, as indicated in the margin.

$$\begin{array}{r|lcl} r & \underline{N} & & \\ r & \underline{Q_1} & \text{rem.} & p_0 \\ r & \underline{Q_2} & \text{"} & p_1 \\ r & \underline{Q_3} & \text{"} & p_2 \\ & \ldots & \ldots & \ldots \\ r & \underline{Q_{n-1}} & \text{"} & p_{n-2} \\ & p_n & \text{"} & p_{n-1} \end{array}$$

Then from the nature of division we have

$$N = Q_1 r + p_0, \quad Q_1 = Q_2 r + p_1, \quad Q_2 = Q_3 r + p_2, \text{ etc.}$$

Therefore

$$\begin{aligned} N &= Q_1 r + p_0 \\ &= Q_2 r^2 + p_1 r + p_0 \\ &= Q_3 r^3 + p_2 r^2 + p_1 r + p_0 \\ &\quad \ldots \ldots \ldots \\ &= p_n r^n + p_{n-1} r^{n-1} \ldots . + p_2 r^2 + p_1 r + p_0, \end{aligned}$$

which is the required number. When figures are used the radix

and signs of addition are omitted, and the last quotient followed by the several remainders are written consecutively.

Ex. 1.—Express 3824 in the scale whose radix is 7.

Dividing continually by 7, the quotients and remainders are as follows:

$$\begin{array}{r|l}
7 & 3824 \\ \hline
7 & 546 \quad \text{rem. } 2 \\ \hline
7 & 78 \quad \text{`` } 0 \\ \hline
7 & 11 \quad \text{`` } 1 \\ \hline
 & 1 \quad \text{`` } 4
\end{array}$$

The required number is 14102.

Ex. 2.—Change 31247 from scale of eight to the common scale.

First Solution.

$$\begin{array}{r}
31247 \\
8 \\ \hline
25 \\
8 \\ \hline
202 \\
8 \\ \hline
1620 \\
8 \\ \hline
12967
\end{array}$$

Second Solution.

$$\begin{array}{r|l}
t & 31247 \\ \hline
t & 2420 \ldots 7 \\ \hline
t & 201 \ldots 6 \\ \hline
t & 14 \ldots 9 \\ \hline
 & 1 \ldots 2
\end{array}$$

Result in each case, 12967.

The reason for the work in the first solution will be perceived by writing the number with the various powers of the radix thus:

$$3(8^4) + 8^3 + 2(8^2) + 4(8) + 7.$$

Now,
$$3(8^4) + 8^3 = (3 \times 8 + 1)8^3 = 25(8^3),$$
$$25(8^3) + 2(8^2) = (25 \times 8 + 2)8^2 = 202(8^2), \text{ etc.}$$

The division in the second solution is performed in the scale of eight; the reasoning is the same as that of the preceding example.

Also, Ex. 1 may be solved by *multiplication*, like the first solution of Ex. 2. The examples which follow should be solved by both methods.

EXERCISE XIII.

1. Add the following numbers, which are expressed in the nonary scale: 32078, 4135, 2057, 38725.

2. From 20100431 take 14034324 in the quinary scale.

3. Multiply 372*t*63 by 38 in the undenary scale.

4. Divide 42765236 by 7 in the octenary scale.

5. Divide 32*te*094 by 11 in the duodenary scale.

6. Divide 30102112 by 1323 in the quaternary scale.

7. From 2061203 take 1626156 in the septenary scale, and multiply the difference by 506.

8. Find the G. C. M. of 323345 and 502341 in the senary scale.

9. Find the L. C. M. of 17, 22, 28, 33, 37, 41 in the undenary scale.

10. Express a million in the nonary scale.

11. Change 320784*t* from the scale of eleven to the common scale.

12. Change 8032765 from the nonary to the septenary scale.

13. Multiply 4541 octenary by 21301 quaternary, extract the square root of the product, and give the result in the septenary scale.

14. Extract the square root of 1100000100001 in the binary scale.

15. Show that the numbers 121, 144, 1234321 are perfect squares in any scale whose radix is greater than 4.

16. Of the weights 1, 2, 4, 8, etc., lbs., which must be taken to weigh 897 lbs.? 1523 lbs.?

17. Of the weights 1, 3, 9, 27, etc., lbs., which must be taken to weigh 1852 lbs., one of each kind only to be taken, but to be placed in whichever scale is necessary?

18. In what scale is 182 represented by 222?

19. Find the scale in which 519 is the square of 23.

20. Find the scale in which the product of 32 and 25 is 1163.

21. Show that 1367631 is a perfect cube in every scale whose radix is greater than 7.

22. Find the scale in which 12736 is represented by 30700.

23. In what scale is 511197 denoted by 1746335?

24. Add the following fractions in the scale of eight:

$$\frac{3}{4}, \frac{7}{10}, \frac{13}{20}, \frac{7}{14}, \frac{5}{30}.$$

25. A rectangle is 13 ft. $6\frac{7}{12}$ in. long and 10 ft. 4 in. wide; find its area by multiplication in the scale of twelve.

26. Find by division in the scale of twelve the height of a right-solid containing 282 cu. ft. 705 cu. in., whose base contains 24 sq. ft. 5 sq. in.

RADIX FRACTIONS.

137. Radix Fractions are a series of fractions whose denominators are successive powers of the radix of the scale, and whose numerators are each less than the radix. Radix fractions in any scale correspond to decimals in the common scale, and are distinguished from integers in the same way, by being preceded by a point, which may be called the radix point.

Thus, in the octenary scale,

$$237 \cdot 6574 = 2(8^2) + 3(8) + 7 + \frac{6}{8} + \frac{5}{8^2} + \frac{7}{8^3} + \frac{4}{8^4};$$

but since in this scale *seven* is the highest number expressed by one symbol, the above must be written

$$237\cdot6574 = 2(10)^2 + 3(10) + 7 + \frac{6}{10} + \frac{5}{10^2} + \frac{7}{10^3} + \frac{4}{10^4},$$

since the radix is always expressed by 10.

138. *To transform a given fraction into radix fractions in any proposed scale.*

Let $\frac{m}{n}$ be the given fraction and r the radix of the proposed scale.

Multiply m by r and divide the product by n; let q_1 be the quotient and r_1 the remainder. Multiply r_1 by r and divide by n; let q_2 be the quotient and r_2 the remainder, and so on. The quotients q_1, q_2, etc., will be the numerators of the radix fractions required.

For
$$\frac{mr}{n} = q_1 + \frac{r_1}{n}, \qquad (1)$$

$$\frac{r_1 r}{n} = q_2 + \frac{r_2}{n}, \qquad (2)$$

$$\text{etc.} = \text{etc.}$$

Therefore
$$\frac{m}{n} = \frac{q_1}{r} + \frac{r_1}{nr}$$

$$= \frac{q_1}{r} + \frac{q_2}{r^2} + \frac{r_2}{nr^2}, \text{ substituting from (2),}$$

$$= \frac{q_1}{r} + \frac{q_2}{r^2} + \frac{q_3}{r^3} + \dots$$

which are the radix fractions required.

139. In the preceding Art. the fraction is supposed to be in its lowest terms. If the denominator n contains no factor except factors of r, the process will terminate, and the number of divisions may be known by considering what power of r is necessary to cancel all the factors of n. If n contains any factor

which is not a factor of r, the process will never terminate; but, since the remainders must each be less than n, their number cannot exceed $n-1$ before some previous remainder recurs, and then the series of quotients will be repeated continually in regular succession. Thus any fraction can be expressed in a finite series of radix fractions, or in a series in which a group of quotients constantly recur, the number of repeating quotients never being greater than the number of units in the denominator of the factor, minus one.

Ex.—Transform $\frac{5}{13}$ from the common scale to a series of radix fractions in the scale of seven.

First Solution.

```
     5
     7
13)35(2
   26
    9
    7
13)63(4
   52
   11
    7
13)77(5
   65
   12
    7
13)84(6
   78
```

Second Solution.

```
16)5·000(·2456
   3 5
   1 20
   1 03
     140
     122
      150
      141
        6
```

Result in each case, ·2456 . . . .

The first solution needs no explanation. In the second method we first change 13 from the decimal scale to its equivalent, 16, in the scale of seven; next place a cipher after the numerator, which is equivalent to multiplying by the radix 7 (which must now be expressed by 10), and divide by the denominator, being careful to perform the whole operation in the new scale.

140. If, in reducing a proper fraction in its lowest terms to a series of radix fractions, a remainder occurs equal to the difference between the numerator and denominator, one-half the recurring period has been found, and the remainder may be obtained by subtracting in order the digits already found from the radix, minus one.

With the notation of Art. 138 suppose a remainder $n-m$ occurs.

Then $$\frac{(n-m)r}{n} = \frac{nr - nq_1 - r_1}{n} = r - 1 - q_1 + \frac{n - r_1}{n},$$

which shows that the next quotient is obtained by subtracting the first one, q_1, from $r-1$, and the remainder, $n-r_1$, being of the same form as $n-m$, the next quotient must also be found by subtracting q_2 from $r-1$, and so on. We thus double the number of digits already found.

In the Ex. of Art. 139, if the process be continued two places further the remainder 8 ($=11$ in second solution) occurs, and it will be found that the succeeding figures are obtained by subtracting those already found from 6. The complete result is

$$\frac{5}{16} = \cdot\dot{2}4563142103\dot{5},$$

both sides of the equality being expressed in the scale of 7. In this example it may be observed that the number of figures repeating is one less than the number of units in the denominator, but in the common scale the number repeating is only half as great.

141. *The difference between any number and the sum of its digits is divisible by $r-1$ where r is the radix of the scale in which the number is expressed.*

Let N denote the number, a, b, c .... the digits in order beginning with the units, S the sum of the digits.

Then $$N = a + br + cr^2 + dr^3 + \dots .$$
$$S = a + b + c + d + \dots .$$

$$\therefore\ N - S = b(r-1) + c(r^2-1) + d(r^3-1) + \dots\dots$$
$$= (r-1)\{b + c(r+1) + d(r^2+r+1) + \dots\dots\}$$

Therefore $r-1$ is a factor of $N-S$.

Cor. 1.—Since $r-1$ divides $N-S$ exactly, N and S must leave the same remainder when separately divided by $r-1$.

Therefore any number in scale r is divisible by $r-1$ when the sum of its digits is divisible by $r-1$.

This gives a convenient test whether a number in the common scale is divisible by 9.

Cor. 2.—The difference between two numbers which consist of the same digits is divisible by $r-1$.

For let N_1 and N_2 denote the numbers, S the sum of the digits in either case; then since $N_1 - S$ and $N_2 - S$ are each divisible by $r-1$, therefore $(N_1 - S) - (N_2 - S)$ or $N_1 - N_2$ is divisible by $r-1$.

142. *If the difference between the sum of the digits in the odd and the even places is a multiple of $r+1$, the whole number is divisible by $r+1$ where r is the radix of the scale in which the number is expressed.*

Let N denote the number, a, b, c .... the digits in order beginning with the units, D the difference between the sums of the two sets of digits.

Then $$N = a + br + cr^2 + dr^3 \dots\dots$$
$$D = a - b + c - d + \dots\dots$$
$$\therefore\ N - D = b(r+1) + c(r^2-1) + d(r^3+1) + \dots\dots$$
$$= (r+1)\{b + c(r-1) + d(r^2-r+1) + \dots\dots\}$$
$$= \text{a multiple of } r+1.$$

Therefore if D is a multiple of $r+1$, so also is N. This proposition gives a convenient test whether a number in the common scale is divisible by 11.

143. If the sums of the digits of two numbers in the common scale be separately divided by 9, and if the product of the two remainders again be divided by 9, the last remainder will be the same as that obtained by dividing by 9 the sum of the digits of the product of the original numbers.

Let N_1 and N_2 denote the numbers, r_1, r_2 the remainders obtained by dividing the sums of their digits by 9; then r_1, r_2 are also the remainders when N_1, N_2 are divided by 9 (Art. 141, Cor. 1). Let q_1, q_2 be the quotients.

Then $$N_1 = 9q_1 + r_1,$$

and $$N_2 = 9q_2 + r_2;$$

$$\therefore\ N_1N_2 = 81q_1q_2 + 9(q_1r_2 + q_2r_1) + r_1r_2$$
$$= \text{a multiple of } 9 + r_1r_2.$$

Therefore the remainder, when r_1r_2 is divided by 9, is the same as when the product N_1N_2 is divided by 9; and this is the same as that obtained by dividing the sum of the digits by 9.

The above is called the "*Rule for casting out the nines.*" It will be observed that the rule fails to detect any error which does not affect the sum of the digits in the result, or which changes their sum by a multiple of 9.

EXERCISE XIV.

1. Transform $\frac{15}{16}$ from the common scale to radix fractions in the scales of four, six, eight and twelve.

2. Transform $\frac{11}{12}$, $\frac{5}{10}$, $\frac{3}{11}$ from the scale of six into ordinary decimals.

3. Transform $\cdot 15625$ and $\cdot 208\dot{3}$ from the common scale to scale six.

4. Transform $\cdot t0t2$ from the scale of eleven to ordinary decimals.

5. Transform 38276·375 from the scale of nine to the scale of twelve.

6. What fractions are equivalent to ·224 and $\cdot 3\dot{3}14\dot{6}$ in the scales of twelve and eight respectively?

7. Find the value of $\cdot\dot{6}$ in scale seven; of $\cdot 1\dot{5}$ in scale eight; of $\cdot\dot{9}\dot{1}$ in scale eleven; and of $\cdot t\dot{9}72\dot{4}$ in scale twelve.

8. Transform $t38t07\cdot\dot{4}7\dot{9}$ from the scale of eleven to the scale of seven.

9. Transform *tet·ee* and *te·tee* from scale twelve to scale eleven.

10. In what scales can $\dfrac{5}{48}$ be exactly expressed by finite radix fractions?

11. In what scale is ·75 of the common scale correctly represented by ·11?

12. In what scale is ·02133252 of scale six correctly represented by ·0508?

13. Transform ·111101011 from the binary to the octenary scale. Why does the result contain just one-third as many digits as the original number? How many digits would be required in scale four?

14. Prove that in the common scale any number is divisible by 4 providing the number formed by its last two digits is divisible by 4. Extend this principle so as to furnish a test for the divisibility of numbers by 8, 16, etc.

15. Give tests for the divisibility of numbers in the octenary scale by four and by sixteen.

16. Find a number of two digits such that its digits are reversed by adding 27, or by adding 9 and transforming into the septenary scale.

17. If any fraction having pq for its denominator be reduced to radix fractions, the number of digits which repeat cannot exceed either $p(q-1)$ or $q(p-1)$.

18. Prove that in every scale whose radix is greater than 8 the number represented by 11088 is divisible by that represented by 12; that the first figure of the quotient is one less than the radix; and that the last two digits are in every case the same.

19. Prove that any number expressed by four digits in the common scale is divisible by 7 providing the first and last digits are equal, and the hundreds digit is twice the tens digit.

20. If a number in the common scale is divisible by 3, the numbers expressed by the same digits in the scales of four, seven and thirteen are also divisible by 3.

21. In a scale whose radix is odd, any number and the sum of its digits are both odd or both even.

22. If S_1 be the sum of the digits of a number, N, expressed in the septenary scale, and $2S_2$ the sum of the digits of $2N$ expressed in the same scale, then the difference between S_1 and S_2 is a multiple of 3.

23. Prove that the square of $rrrr$ in the scale of s is $rrrq0001$, where q, r, s are any three consecutive integers.

24. In the scale of notation whose radix is r, show that the number $(r^2-1)(r^n-1)$, when divided by $r-1$, will give a quotient with the same digits in the reverse order.

25. If from any number expressed in the nonary scale is subtracted the sum of every third digit beginning with the units, twice the sum of every third digit beginning with the tens, and four times the sum of the remaining digits, the remainder is divisible by 7.

26. Show that any number of six digits in the common scale, whose first and fourth, second and fifth, third and sixth digits are alike, is divisible by 7, 11 and 13.

27. A certain fraction is correctly represented by $\cdot\dot{2}\dot{1}$ in the scale of x, by $\cdot\dot{2}\dot{7}$ in the scale of y, and by $\cdot 5$ in the scale of $x+y$; express the fraction as an ordinary decimal.

28. The digits of a number are added, the digits of this sum are added, and so on until the last sum is a single digit. If this operation be performed upon several numbers, and then the same operation upon the resulting single digits, the final result will be the same as that obtained by performing the same operation upon the sum of the original numbers.

29. If $\dfrac{1}{(r-1)^2}$ be reduced to radix fractions in the scale of r, show that the period which repeats is composed of zero followed by the digits in order up to $r-1$, omitting the digit $r-2$.

30. If p_0, p_1, p_2.... be the digits of any number beginning with the units, prove that the number is divisible (1) in the common scale by 8 if $p_0+2p_1+4p_2$ is divisible by 8; (2) in the scale of twelve by 8 if $4p_1+p_0$ is divisible by 8; and by 2, 3 or 6 providing p_0 is so divisible. Give similar tests for the divisibility of numbers in the scale of twelve by sixteen, eighteen, twenty-four and seventy-two.

31. If the digits of any number in the common scale be divided into groups of six digits each beginning with the units, and if the digits in order of each group be multiplied by 1, 3, 2, 6, 4, 5 respectively, and the sum of the products be subtracted from the given number, the remainder will be divisible by 7. Give a similar theorem when 13 is substituted for 7.

CHAPTER X.

SQUARE AND CUBE ROOTS, AND SURDS.

SQUARE ROOT OF NUMBERS.

144. Practical rules for the extraction of the square and cube roots of numbers are given in all works on arithmetic, but the reasoning employed, being algebraical, is not suitable for students at that stage of their studies. We shall now, therefore, deduce the reason for the ordinary rules for extracting the square and cube roots from the methods given (Part I., Chapter XI.) for the extraction of the corresponding roots of algebraical expressions. Examples of whole numbers only have been given, but the principles are equally applicable to decimals.

145. The integral part of the square root of a number less than 100 consists of *one* digit; of a number between 100 and 10,000 consists of *two* digits; of a number between 10,000 and 1,000,000 consists of *three* digits, and so on. In other words, the square root of a number consists of one, two, three, etc., digits, according as the number consists of one or two digits, three or four digits, five or six digits, etc. If, then, we divide the digits of the given number into groups of two digits each beginning with the units, the number of groups will give the number of digits in the root. This gives us the highest power of 10 contained in the root, which, multiplied by the largest integer whose square is not greater than the left-hand group, gives a first approximation or is the first figure in the required root.

146. Let N denote a number whose square root is to be found, and let a denote a first approximation found by the last Art., and let x denote the remaining part of the root.

Then $$N = a^2 + 2ax + x^2 \text{ or } N - a^2 = 2ax + x^2,$$

from which equations the value of x must be found.

Now, neglecting x^2, which is considerably less than $2ax$, we get $x = \dfrac{N - a^2}{2a}$. Let x_1 be the first figure of the quotient, followed by the proper number of ciphers; add it to $2a$, multiply the result by x_1 and subtract from $N - a^2$, and we get

$$N - a^2 - 2ax_1 - x_1^2 \text{ or } N - (a + x_1)^2.$$

Denote $a + x_1$ by a_1 and the remaining part of the root by x_2, and proceed as before.

In practical work it will frequently be found that $2ax_1 + x_1^2$ is greater than $N - a^2$; in such cases a smaller integer than x_1 must be taken.

Ex.—Find the square root of 119025.

Dividing the digits into groups of two digits each we see that the root must contain three digits; and since the greatest integer whose square is less than 11 (the left-hand group) is 3, therefore 300 is a first approximation.

Then $$119025 = (300 + x)^2 = 90000 + 600x + x^2,$$

$$\therefore\ 600x + x^2 = 29025 \text{ or } x = 40;$$

then $$(600 + 40) \times 40 = 25600;\ 29025 - 25600 = 3425.$$

Again, $$680x + x^2 = 3425 \text{ or } x = 5,$$

then $$(680 + 5) \times 5 = 3425,$$

which shows that 345 is the root required.

The preceding work may be arranged thus:

$$\begin{array}{rl}
300 & 11\ 90\ 25\,(300+40+5 \\
 & 9\ 00\ 00 \\
300\times 2 = 600 & \overline{2\ 90\ 25} \\
40 & 2\ 56\ 00 \\
\overline{640} & \overline{34\ 25} \\
340\times 2 = 680 & 34\ 25 \\
5 & \\
\overline{685} &
\end{array}$$

The student should compare the above with what precedes, then omit the zeros and arrange the work in the usual way as below, and observe that the final operation is only a convenient arrangement of the process first given.

$$\begin{array}{ll}
 & 119025\,(345 \\
3\ \ldots\ldots & 9 \\
64\ \ldots\ldots & \overline{290} \\
 & 256 \\
685\ \ldots\ldots & \overline{3425} \\
 & 3425
\end{array}$$

147. *When $n+1$ figures of the square root of a number have been found by the ordinary process, n more may be found by dividing the last remainder by twice the root already found, the whole root consisting of $2n+1$ figures.*

Let a denote the part of the root already found, x the part to be found, N the given number.

Then $$N = a^2 + 2ax + x^2;$$

therefore $$N - a^2 = 2ax + x^2$$

and $$\frac{N-a^2}{2a} = x + \frac{x^2}{2a}.$$

Now, $N - a^2$ is the last remainder, and this divided by $2a$, *i.e.*, twice the root already found, gives x the part to be found increased by $\frac{x^2}{2a}$, which we shall show to be less than unity.

Now, a contains $n+1$ digits followed by n digits, $\therefore\ a > 10^{2n}$; and x contains n digits, $\therefore\ x < 10^n$.

Therefore $$\frac{x^2}{2a} < \frac{10^{2n}}{2(10)^{2n}} < \frac{1}{2},$$

which proves the proposition.

If the number is not a complete square the above demonstration fails. But the quotient obtained by this method in all cases differs from the true root by less than a unit in the last digit; it may therefore always be used for finding approximate values of the roots of surds. Similar remarks apply to the theorems of Arts. 151 and 152.

CUBE ROOT OF NUMBERS.

148. The integral part of the cube root of a number less than 1,000 consists of one digit; of a number between 1,000 and 1,000,000, of two digits, etc. Therefore the cube root of a number consisting of one, two or three digits, consists of one digit; of a number consisting of four, five or six digits, consists of two digits, etc. If, then, we divide the digits of a given number into groups of three digits each beginning with the units, the number of groups will give the number of units in the root. This gives us the highest power of 10 contained in the root, which, multiplied by the largest integer whose cube is not greater than the left-hand group, gives a first approximation to the required root.

149. Let N denote a number whose cube root is to be found, and let a be the part found as above, and let x be the remaining part of the root.

Then $$N = a^3 + 3a^2x + 3ax^2 + x^3,$$

or $$N - a^3 = 3a^2x + 3ax^2 + x^3$$

$$= (3a^2 + 3ax + x^2)x.$$

Neglecting the terms $3ax^2 + x^3$, which are less than $3a^2x$, we get $x = \dfrac{N - a^3}{3a^2}$. Let x_1 be the first figure of the quotient, followed by the proper number of ciphers; substitute it for x in the expression, $(3a^2x + 3ax + x^2)x$, and subtract it from the last remainder, viz., $N - a^3$, giving $N - (a + x_1)^3$; for $a + x_1$ write a_1, and proceed as before until there is no remainder, or until the root has been found to the required degree of accuracy.

Should the numerical value of $(3a^2 + 3ax_1 + x_1^2)x_1$ be greater than $N - a^3$, an integer smaller than x_1 must be taken.

150. In practical work the tedious part of the operation consists in calculating the values of the trial divisors, $3a^2$, $3a_1^2$, etc., and of the complete divisors, $3a^2 + 3ax + x^2$, etc. By properly arranging the work their values may be calculated in succession, the value of each being used in finding the value of the following one. The method will be evident from the arrangement of the quantities in the two columns below. The first approximation to the root is denoted by a, and the successive additions to it by b, c, etc., which for distinctness may be called quotients.

First Column.	*Second Column.*
$3a$,	$3a^2$,
$3a + b$,	$3a^2 + 3ab + b^2$,
$3a + 2b$,	$3(a + b)^2$,
$3(a + b) + c$,	$3(a + b)^2 + 3(a + b)c + c^2$,
$3(a + b) + 2c$,	$3(a + b + c)^2$,
$3(a + b + c) + d$.	$3(a + b + c)^2 + 3(a + b + c)d + d^2$.

In the above observe:

(1) The successive quotients, a, b, c, etc., are found by dividing the successive remainders by the trial divisors, $3a^2$, $3(a + b)^2$, etc.

(2) The successive quantities in the first column are formed by adding b, b, $b+c$, c, $c+d$, d, $d+e$, etc., each quotient being added three times.

(3) Each quantity in the second column is formed by multiplying the corresponding quantity in the first column by the last quotient, and adding the result to the preceding quantity in the second column.

Ex.—Extract the cube root of 12288010982976.

		12288010982976(23076
		8
6	12	4288
63	1389	4167
66	1587	121010982
6907	15918349	111428443
6914	15966747	9582539976
69216	1597089996	9582539976

EXPLANATION.—The digits of the given number are divided into groups of three figures each for reasons already explained. The greatest integer whose cube is not greater than 12 (the group on the left) is 2; therefore 2 is the first figure of the root. Cubing 2, subtracting from 12, and bringing down the next period, we have 4288 the first remainder. Three times 2 gives 6, the first number in the first column; multiplying the 6 by 2 we get 12, the first number of the second column and the first trial divisor. Rejecting the last two figures, 88, of the remainder we get 3 for quotient, the second figure of the root. The various succeeding numbers in the two columns correspond exactly to the algebraical quantities in the columns formerly given, the letters a, b, c, d being replaced by 20000, 3000, 70, 6, the ciphers being omitted for brevity. The student should write out the work in full, then remove the ciphers, when the remaining figures will be found to be those given above, thus showing the reason for the given arrangement.

151. *When $n+2$ figures of the cube root of a number have been found by the ordinary process, then the integral part of the quotient obtained by dividing the last remainder by three times the square of the root already found will be the remaining part of the root required, the whole root consisting of $2n+2$ figures.*

Let N be the given number, a the part of the root found, x the part to be found.

Then
$$N = (a+x)^3$$
$$= a^3 + 3a^2x + 3ax^2 + x^3.$$
$$\therefore\ N - a^3 = 3a^2x + 3ax^2 + x^3,$$
and
$$\frac{N-a^3}{3a^2} = x + \frac{x^2}{a}\left(\frac{3a+x}{3a}\right).$$

Now,
$$\frac{3a+x}{3a} < \frac{4}{3} \text{ because } x < a,$$
and
$$\frac{x^2}{a} < \frac{(10)^{2n}}{(10)^{2n+1}} < \frac{1}{10},$$

since x contains n digits and a contains $2n+2$ digits.

Therefore $\frac{x^2}{a}\left(\frac{3a+x}{3a}\right)$ is less than a unit, and x, the integral part of the quotient, is the root required.

152. *When n figures of the cube root of a given number have been found by the ordinary process, $2(n-1)$ more figures may be obtained by dividing the product of the last remainder and the part already found by the sum of the given number and twice the cube of the root already found.*

With the notation of the previous Art. we have

$$\frac{(N-a^3)a}{N+2a^3} = \frac{(3a^2x+3ax^2+x^3)a}{3a^3+3a^2x+3ax^2+x^3} = x\left\{1 - \frac{x^2(2a+x)}{N+2a^3}\right\}.$$

Therefore, if we take $\frac{(N-a^3)a}{N+2a^3}$ instead of the true value, x, we make an error of $\frac{x^3(2a+x)}{N+2a^3}$.

Now, $$N>a^3 \text{ and } a>x;$$

therefore $$\frac{x^3(2a+x)}{N+2a^3}<\frac{x^3(2a+x)}{3a^3}<\frac{x^3}{a^2}.$$

If a contains n digits and x contains r digits,

then $$a>(10)^{n+r-1} \text{ and } x<(10)^r.$$

Therefore $$\frac{x^3}{a^2}<\frac{(10)^{3r}}{(10)^{2(n+r-1)}}<\frac{(10)^r}{(10)^{2(n-1)}}.$$

And the error is less than a unit if r is not greater than $2(n-1)$.

NOTE.—In practical work a^3 is found at once by subtracting the last remainder from the given number.

Ex.—To find the cube root of 7.

By the ordinary method we find the first three figures to be 1·91, and the remainder, ·032129.

Then $$a^3=7-\cdot032129=6\cdot967871,$$

and $$\frac{\cdot032129\times1\cdot91}{7+2(6\cdot967871)}=\cdot002931.$$

Therefore $$\sqrt[3]{7}=1\cdot912931.$$

SURDS.

153. The most important properties of surds have already been explained (Part I., Chapter XIII.); we now discuss a few more complicated examples, chiefly in connection with the extraction of the square and cube roots of surd expressions.

154. The square root of the sum of two quadratic surds may sometimes be expressed as the sum of two fourth roots.

For the expression $\sqrt{a^2c} + \sqrt{b}$ may be written

$$\sqrt{c}\left(a + \sqrt{\frac{b}{c}}\right);$$

and if $a^2 - \frac{b}{c}$ is a perfect square, the root of this quantity may be expressed in the form

$$\sqrt[4]{c}\,(x + \sqrt{y}) \text{ or } \sqrt[4]{cx^2} + \sqrt[4]{cy^2}.$$

Ex. Find the square root of $\sqrt{27} + \sqrt{24}$.

$$\sqrt{27} + \sqrt{24} = \sqrt{3}(3 + 2\sqrt{2}) = \sqrt{3}(\sqrt{2} + 1)^2.$$

Therefore $\sqrt[4]{3}(\sqrt{2} + 1) = \sqrt[4]{12} + \sqrt[4]{3}$ is the root required.

155. The square of the sum of three surds consists of four terms, viz., a rational quantity and three surds. Hence we may sometimes find the square root of a quantity of the form

$$a + \sqrt{b} + \sqrt{c} + \sqrt{d}.$$

Assume $\sqrt{a + \sqrt{b} + \sqrt{c} + \sqrt{d}} = \sqrt{x} + \sqrt{y} + \sqrt{z}.$

Squaring,

$$a + \sqrt{b} + \sqrt{c} + \sqrt{d} = x + y + z + 2\sqrt{xy} + 2\sqrt{yz} + 2\sqrt{zx}.$$

If, then, values for x, y, z can be found such that

$$2\sqrt{xy} = \sqrt{b}, \quad 2\sqrt{yz} = \sqrt{c}, \quad 2\sqrt{zx} = \sqrt{d},$$

and if the values thus found also satisfy $x + y + z = a$, we shall evidently have the root required.

Ex.—Extract the square root of $12 - 4\sqrt{3} - 2\sqrt{15} + 4\sqrt{5}$.

Assume $\sqrt{12 - 4\sqrt{3} - 2\sqrt{15} + 4\sqrt{5}} = \sqrt{x} - \sqrt{y} + \sqrt{z}$,

then

$$12 - 4\sqrt{3} - 2\sqrt{15} + 4\sqrt{5} = x + y + z - 2\sqrt{xy} - 2\sqrt{yz} + 2\sqrt{zx}.$$

Put $\quad 2\sqrt{xy} = 4\sqrt{3}, \quad 2\sqrt{yz} = 2\sqrt{15}, \quad 2\sqrt{zx} = 4\sqrt{5}$,

or $\quad xy = 12, \quad yz = 15, \quad zx = 20$,

from which we easily get $x = 4$, $y = 3$, $z = 5$; and these values also satisfy $x + y + z = 12$, therefore $2 - \sqrt{3} + \sqrt{5}$ is the root required.

156. In the preceding Art., if the values found for x, y, z do not satisfy the equation, $x + y + z = a$, it would be erroneous to infer that the given expression has no square root. The correct inference is that it has not a square root of the assumed form; it may have a root of a different form. For example, consider the expression, $12 + 8\sqrt{2} + 6\sqrt{3} + 4\sqrt{6}$.

Proceeding as before we obtain the equations,

$$2\sqrt{xy} = 8\sqrt{2}, \quad 2\sqrt{yz} = 6\sqrt{3}, \quad 2\sqrt{zx} = 4\sqrt{6},$$

which are satisfied by $x = 5\frac{1}{3}$, $y = 6$, $z = 4\frac{1}{2}$. But these values do not satisfy $x + y + z = 12$, therefore the square root of the given expressions is not of the form $\sqrt{x} + \sqrt{y} + \sqrt{z}$. The correct root is $1 + \sqrt{2} + \sqrt{3} + \sqrt{6}$; but no direct process can be given for obtaining the root in such cases. It will be instructive for the student to write out the square of each of the expressions,

$$x + \sqrt{y} + \sqrt{z} + \sqrt{yz}, \quad \sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{xyz},$$

$$m + \sqrt{xy} + \sqrt{yz} + \sqrt{zx},$$

and to observe that the result in each case is of the same form as that of the preceding Art. But if we attempt to obtain the root of a numerical example by using any one of these results, we shall find the resulting equations too difficult for solution.

10

157. *If* $\sqrt[3]{a+\sqrt{b}}=x+\sqrt{y}$, *then will* $\sqrt[3]{a-\sqrt{b}}=x-\sqrt{y}$.

For by cubing we obtain

$$a+\sqrt{b}=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y}.$$

Equating the rational, and also the irrational, parts, we have

$$a=x^3+3xy, \quad \sqrt{b}=3x^2\sqrt{y}+y\sqrt{y}.$$

Therefore $\qquad a-\sqrt{b}=x^3-3x^2\sqrt{y}+3xy-y\sqrt{y},$

or $\qquad \sqrt[3]{a-\sqrt{b}}=x-\sqrt{y}.$

Similarly it may be shown that if

$$\sqrt[n]{a+\sqrt{b}}=x+\sqrt{y}, \text{ then } \sqrt[n]{a-\sqrt{b}}=x-\sqrt{y},$$

where n is any positive integer.

158. *To extract the cube root of a binomial quadratic surd.*

Since $\qquad (x+\sqrt{y})^3=x^3+3xy+(3x^2+y)\sqrt{y},$

and $\qquad (\sqrt{x}+\sqrt{y})^3=(x+3y)\sqrt{x}+(y+3x)\sqrt{y},$

we see that the cube of a binomial quadratic surd is a quadratic surd of the *same form*. We therefore reduce the given surd to its simplest form, and assume its root to be a similar surd.

1. If one term be rational:

Assume $\qquad \sqrt[3]{a+n\sqrt{b}}=x+y\sqrt{b}. \qquad (1)$

Then $\qquad \sqrt[3]{a-n\sqrt{b}}=x-y\sqrt{b}, \qquad$ Art. 157

therefore $\qquad \sqrt[3]{a^2-n^2b}=x^2-by^2. \qquad (2)$

Cubing (1), and equating the rational parts, we get

$$x(x^2+3by^2)=a. \qquad (3)$$

and from (2), $\qquad x^2-by^2=c, \qquad (4)$

where $\qquad c^3=a^2-n^2b.$

2. If both terms be surds:

Assume $\sqrt[3]{m\sqrt{a}+n\sqrt{b}}=x\sqrt{a}+y\sqrt{b}$,

Then, as before, $x(ax^2+3by^2)=m$, (5)

and $ax^2-by^2=c$, (6)

where $c^3=m^2a-n^2b$.

Now, if the original surd is an exact cube, and its coefficients are positive integers, x and y must also be positive integers, and therefore a^2-n^2b and m^2a-n^2b must be exact cubes, since each is equal to an integral quantity. If the coefficients are not integral they may be made integral by multiplying through by the proper factor, and then the root must be divided by the root of this factor.

The values of x and y must be found from (3) and (4), or from (5) and (6), by trial; but since they are positive integers, in most cases this may easily be done. The numerical examples which follow show the best method of proceeding.

Ex. 1.—Find the cube root of $207+94\sqrt{5}$.

Assume $\sqrt[3]{207+94\sqrt{5}}=x+y\sqrt{5}$.

Then $\sqrt[3]{207-94\sqrt{5}}=x-y\sqrt{5}$,

from which $x^2-5y^2=\sqrt[3]{(207)^2-5(94)^2}$,

or $x^2-5y^2=-11$, (1)

and $x(x^2+15y^2)=207$. (2)

From (1), $x^2=5y^2-11$. Giving y the values 1, 2, etc., in succession, we find $y=2$, $x=3$ satisfies this equation and also equation (2); therefore $3+2\sqrt{5}$ is the root required.

Ex. 2.—Find the cube root of $9217-4122\sqrt{5}$.

The equations are:

$$x^2-5y^2=-11, \qquad (1)$$

$$x(x^2+15y^2)=9217. \qquad (2)$$

The values $y=2$, $x=3$ satisfy (1) as before, but do not satisfy (2). Giving y the values 3, 4, etc., we find $y=6$ gives a complete square, 169, for the value of x^2; therefore $13-6\sqrt{5}$ is the root required.

Ex. 3.—Extract the cube root of $430\sqrt{2}+324\sqrt{3}$.

Assume $\qquad \sqrt[3]{430\sqrt{2}+324\sqrt{3}}=x\sqrt{2}+y\sqrt{3}.$

Then $\qquad \sqrt[3]{430\sqrt{2}-324\sqrt{3}}=x\sqrt{2}-y\sqrt{3},$

therefore
$$2x^2-3y^2=\sqrt[3]{(430\sqrt{2})^2-(324\sqrt{3})^2}$$
$$=38,$$

and
$$x(2x^2+9y^2)=430.$$

Giving y the values 1, 2, etc., until an integral value is also obtained for x, we find $y=2$, $x=5$ satisfies both equations; therefore $5\sqrt{2}+2\sqrt{3}$ is the root required.

159. The student's progress in many parts of mathematics, especially in the solution of equations and in Trigonometry, will be much facilitated by a thorough knowledge of surds. We therefore give a large collection, chiefly selected from examples which have presented themselves in practical work.

EXERCISE XV.

1. Find the square root of 10, $\dfrac{3}{4}$ and 3·1415926536, each to ten decimal places.

2. Find the cube root of 2, ·2 and 1·9098593172, each to ten significant figures.

3. Find the value of $\dfrac{1}{4}\sqrt{10+2\sqrt{5}}$ and of $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$, each to seven places of decimals.

4. Find the square root of

$$8\sqrt{2}+2\sqrt{30},\ 7\sqrt{3}-12,\ \frac{3}{\sqrt{2}}-2,\ \text{and}\ n\sqrt{m}-2m\sqrt{n-m}.$$

5. Find the square root of

$$16-2\sqrt{20}-2\sqrt{28}+\sqrt{140}\ \text{and of}\ 40+12\sqrt{6}+8\sqrt{10}+6\sqrt{15}.$$

6. Find the square root of

$$21+3\sqrt{8}-6\sqrt{3}-6\sqrt{7}-\sqrt{24}-\sqrt{56}+2\sqrt{21}.$$

7. Extract the cube root of

$$7+5\sqrt{2},\ 72-32\sqrt{5}\ \text{and}\ 135\sqrt{3}-87\sqrt{6}.$$

8. Simplify $\{1351-780\sqrt{3}\}^{\frac{1}{3}}-\{26+15\sqrt{3}\}^{-\frac{2}{3}}$.

9. Divide

$$\sqrt{3}+3\ \text{by}\ 3\sqrt{3}+5\ \text{and}\ x-x^2+2x\sqrt{1-x}\ \text{by}\ 1+x-\sqrt{1-x}.$$

10. Simplify $(x-1+\sqrt{2})(x-1-\sqrt{2})(x+2+\sqrt{3})(x+2-\sqrt{3})$.

11. Simplify $\dfrac{1-x\sqrt{3}}{x(x+\sqrt{3})}+\dfrac{(1-x\sqrt{3})(1+x\sqrt{3})}{(x+\sqrt{3})(x-\sqrt{3})}+\dfrac{1+x\sqrt{3}}{x(x-\sqrt{3})}$.

12. Simplify

$$(\sqrt{5}+\sqrt{3}+\sqrt{2}+1)^2+(\sqrt{5}+\sqrt{3}-\sqrt{2}-1)^2$$
$$+(\sqrt{5}-\sqrt{3}+\sqrt{2}-1)^2+(\sqrt{5}-\sqrt{3}-\sqrt{2}+1)^2.$$

13. Express $\dfrac{(\sqrt{3}+\sqrt{5})(\sqrt{5}+\sqrt{2})}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ as an equivalent fraction with a rational denominator.

14. Show that $\dfrac{\sqrt{3}+1}{2\sqrt{2}-\sqrt{3}+1}=(\sqrt{2}+\sqrt{3})(\sqrt{2}-1)$.

15. Simplify $1-\dfrac{y^2}{x^2+y^2+x\sqrt{x^2+y^2}}$.

16. Simplify $\sqrt[3]{2\sqrt{2\sqrt[3]{2}}} \times \sqrt[3]{2\sqrt[3]{2\sqrt{2}}} \div \sqrt{2\sqrt[3]{2\sqrt[3]{2}}}$.

17. Show that $\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}} = \{8 - 2\sqrt{10+2\sqrt{5}}\}^{\frac{1}{2}}$,

and that $$\sqrt[3]{\frac{1}{3}} + \sqrt[3]{\frac{2}{3}} = \sqrt[3]{\sqrt[3]{4}+\sqrt[3]{2}+1} = \sqrt[3]{\frac{1}{\sqrt[3]{2}-1}}.$$

18. Find the continued product of the six factors,

$$x^2 - \frac{\sqrt{3}+1}{\sqrt{2}}x + 1, \quad x^2 - \frac{\sqrt{3}-1}{\sqrt{2}}x + 1, \quad x^2 + x\sqrt{2} + 1,$$

$$x^2 - x\sqrt{2} + 1 \qquad x^2 + \frac{\sqrt{3}+1}{\sqrt{2}}x + 1, \quad x^2 + \frac{\sqrt{3}-1}{\sqrt{2}}x + 1.$$

19. Multiply $x^2 - (\sqrt[3]{2} - 1)x + \sqrt[3]{4} + \sqrt[3]{2} + 1$ by $x + \sqrt[3]{2} - 1$.

20. Divide $2x^3 - 6x + 5$ by $x\sqrt[3]{2} + \sqrt[3]{4} + 1$.

21. Simplify $\dfrac{\sqrt{3-\sqrt{5}}}{\sqrt{2}+\sqrt{7-3\sqrt{5}}}$ and $\dfrac{2\sqrt{2+\sqrt{3}}}{4-\sqrt{6-4\sqrt{2}}}$.

22. Simplify $\dfrac{(48\frac{1}{16} + \frac{23}{2}\sqrt{15})^{\frac{3}{4}} + (48\frac{1}{16} - \frac{23}{2}\sqrt{15})^{\frac{3}{4}}}{\sqrt{20}}$.

23. Show that $\sqrt{a^2 + \sqrt[3]{a^4b^2}} + \sqrt{b^2 + \sqrt[3]{a^2b^4}} = (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

24. Simplify $\dfrac{\sqrt{16-6\sqrt{7}}}{\sqrt{3}+\sqrt[4]{7}}$ and $2\sqrt[4]{24\sqrt[3]{18}} \div \sqrt[6]{\frac{2}{3}} \cdot \sqrt{2\sqrt{12}}$.

25. Find the value of $x^3 - 3\sqrt[3]{2}\,.\,x$ when $x = \dfrac{\sqrt{3}+1}{\sqrt[3]{2}}$.

26. Find the value of $x^3 + 3qx$ when

$$x = \sqrt[3]{r + \sqrt{r^2+q^3}} + \sqrt[3]{r - \sqrt{r^2+q^3}}.$$

27. Simplify $(1+\sqrt{2}-\sqrt{3})\sqrt{2-\sqrt{2}} - 2\sqrt{2 - \dfrac{\sqrt{3}+1}{\sqrt{2}}}$.

28. Simplify

$$\frac{2\sqrt{3}(1+\sqrt{3}+\sqrt{6})}{(\sqrt{2}+\sqrt{3})(\sqrt{3}+\sqrt{6})(\sqrt{6}+\sqrt{2})} \text{ and } \frac{\sqrt{26-15\sqrt{3}}}{5\sqrt{2}-\sqrt{38+5\sqrt{3}}}.$$

29. Show that

$$\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)^5 = \frac{5}{64}\{2\sqrt{10-2\sqrt{5}}-\sqrt{10+2\sqrt{5}}\}.$$

30. Extract the cube root of $9ab^2+(b^2+24a^2)\sqrt{b^2-3a^2}$.

31. Show that $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}=\sqrt{2}.$

32. Show that $(\sqrt{3}+\sqrt{2}-1)\sqrt{2+\sqrt{2}}=2\sqrt{2+\dfrac{\sqrt{3}+1}{\sqrt{2}}}$

33. Find the value of $\dfrac{10\sqrt{2}}{\sqrt{18}-\sqrt{3+\sqrt{5}}}-\dfrac{\sqrt{10}+\sqrt{18}}{\sqrt{8}+\sqrt{3-\sqrt{5}}}$ to five places of decimals.

34. Show that

$$\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right)^{\frac{1}{2}}+\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^{\frac{3}{2}}=\frac{2(2-x)}{x^2}(\sqrt{x}-\sqrt{x-x^2}).$$

35. Express $\left(\dfrac{2\sqrt{2}-\sqrt{3}-1}{2\sqrt{2}+\sqrt{3}+1}\right)^{\frac{1}{2}}+\left(\dfrac{2\sqrt{2}-\sqrt{3}+1}{2\sqrt{2}+\sqrt{3}-1}\right)^{\frac{1}{2}}$ as the difference of two simple surds.

36. Simplify

$$\frac{(\sqrt{3}+\sqrt{2}-1)\sqrt{2+\sqrt{2}}}{2\sqrt{4+\sqrt{6}+\sqrt{2}}} \text{ and } \frac{\sqrt[3]{7\cdot 29}-\sqrt[3]{\cdot 15625}}{\sqrt[3]{270}+\sqrt[3]{33\cdot 75}}.$$

37. Find the value of

$$\frac{a^2-b^2}{2(a^2+b^2)}(\sqrt[p]{x}+\sqrt[q]{x}) \text{ when } x=\left(\frac{a+b}{a-b}\right)^{\frac{2pq}{q-p}}$$

CHAPTER XI.

IMAGINARY QUANTITIES.

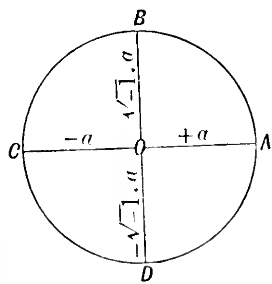
160. From the meaning given to Multiplication the product of two equal factors has been shown to be essentially positive, and the square root of an algebraical expression has been defined to be one of two *equal* factors whose product is the given expression; from which it follows that to speak of the square root of a negative quantity is a contradiction of terms, and is therefore an absurdity. For this reason the terms, "impossible," "imaginary," "not real," have been applied to symbols denoting such contradictory operations. When, however, the proper meaning is attached to the symbol $\sqrt{-1}$, which may be taken as the representative of all the so-called imaginary expressions, it becomes quite as *real* and intelligible as any other symbol whatever. But the words "imaginary," etc., are too firmly fixed in the language of mathematics to be changed, and this is the necessary and sufficient reason for their being retained.

It is customary in mathematical works to *assume* that imaginary quantities are subject to all the operations of elementary algebra without assigning any intelligible meaning to either the symbols of quantity or the operations performed upon them; and this course was adopted in the brief treatment given in Part I. We shall now give a rigorous investigation of the truth of what was there assumed, according to the meaning which we shall assign to the symbols of imaginary quantities and the operations to be performed upon and by them.

161. When a quantity is multiplied by a negative number different from unity, two distinct operations are performed: (1) the

magnitude is increased or diminished, and (2) its relation to some other quantity is changed, *i.e.*, it is changed from *positive* to *negative*, or *vice versa*. For the present leaving out the numerical value, and taking -1 for our multiplier, let us carefully examine its effect. By multiplying by -1, a number denoting cash in hand or money due *to* me is transformed into a number denoting a debt due *by* me; a number denoting time reckoned *after* a given event into a number denoting time *preceding* that event; and a number denoting a distance measured in one direction into a number denoting an equal distance in the opposite direction. Now the question arises, Is it intelligible to speak of performing a *part* of any one of these operations? or, in other words, *Is there any intermediate stage between positive and negative?* In the case of *distance* and *direction* there is; in all other cases there is not; consequently $\sqrt{-1}$ has an intelligible meaning when applied to space, but is unintelligible in connection with any other kind of quantity.

162. Let $ABCD$ be a circle, radius a. Draw the diameters AOC, BOD at right angles to each other; then if OA be denoted by $+a$, OC will be correctly represented by $-a$; therefore OA multiplied by -1 becomes OC. In the process of changing OA into OC, conceive that OA revolves around O, through the semicircle ABC, and consequently passes through the position OB. Now, distance measured in direction OB is neither *positive* nor *negative*, it is the intermediate stage referred to in the last Art. To turn OA through a right angle into the position OB is to perform *half* the operation of multiplying it by -1; for if the operation be repeated upon OB the result is OC, which is the result obtained by multiplying OA by -1. Now, to multiply twice by the square root of a number gives the same result as to multiply once by the number; there-



fore, as a multiplier, the square root of a number bears the same relation to the number itself as the operation of turning a line through *one* right angle bears to the operation of turning it through *two* right angles, which is equivalent to multiplying it by -1. For this reason it is convenient (and reasonable) to define $\sqrt{-1}$ to be *the symbol of the operation of turning a line from its original position through one right angle.*

163. The operations symbolized by $\sqrt{-1}$ may be performed upon the result of a previous operation of the same kind, thus:

$$\sqrt{-1} \,.\, OA = OB, \qquad \sqrt{-1} \,.\, OB = OC,$$
$$\sqrt{-1} \,.\, OC = OD, \qquad \sqrt{-1} \,.\, OD = OA, \text{ etc.}$$

If, now, we denote one of these quantities, OA, by a, and the number of operations performed upon it by an exponent affixed to the operator, we shall have the following results:

$$OB = \sqrt{-1} \,.\, OA = \sqrt{-1} \,.\, a;$$
$$OC = \sqrt{-1} \,.\, OB = (\sqrt{-1})^2 \,.\, a = -a, \text{ since } OC = -OA;$$
$$OD = \sqrt{-1} \,.\, OC = (\sqrt{-1})^3 \,.\, a = -\sqrt{-1} \,.\, a, \text{ since } OD = -OB;$$
$$OA = \sqrt{-1} \,.\, OD = (\sqrt{-1})^4 \,.\, a = +a.$$

Since $(\sqrt{-1})^4 \,.\, a = a$, the symbol $(\sqrt{-1})^4$ in connection with any quantity may be introduced or omitted any number of times without producing any change whatever. This principle enables us to give at once the result of any number of such operations. Thus $(\sqrt{-1})^n \,.\, a = a$, $\sqrt{-1} \,.\, a$, $-a$, or $-\sqrt{-1} \,.\, a$, according as n, when divided by 4, gives 0, 1, 2 or 3 for remainder.

164. It should be observed that it would have been equally correct to assume that $\sqrt{-1}$ as an operator turns a line in a direction opposite to that which we have chosen. Had this been done the symbols representing OD and OB would simply have

been interchanged. The direction chosen, as in the case of ordinary positive and negative quantities, is of no consequence; but when once chosen it must be rigidly adhered to throughout any one investigation. Mathematicians, however, are unanimous in deciding that *the positive direction of rotation shall be contrary to that of the hands of a watch*, which agrees with what has thus far been assumed.

165. Since $\sqrt{-1}$ is the symbol of an operation, it can have no meaning except when taken in connection with a quantity *upon which it can operate;* and since $(\sqrt{-1})^2$ merely denotes a repeated operation, it is also without meaning when standing alone. But $(\sqrt{-1})^2$ as an *operator* is equivalent to -1 in the same capacity; therefore it is convenient to assume $(\sqrt{-1})^2 = -1$, since they are the symbols of equivalent operations. It is also convenient to speak of $\sqrt{-1}$ as a multiplier or factor, of $(\sqrt{-1})^n$ as a power of $\sqrt{-1}$, and of the operations symbolized as multiplication. Again, as $\sqrt{-1}.a$ denotes a line a units in length drawn in a particular direction, so $\sqrt{-1}.1$ should be written to denote a line 1 unit in length drawn in the same direction; but the 1 is always omitted, and $\sqrt{-1}$ is written either as a symbol of operation or as a kind of unit. And since $\sqrt{-1}$ is written to denote one unit, $a.\sqrt{-1}$ is written to denote a units of the same kind; so that when $\sqrt{-1}$ stands first it denotes an operation to be performed on what follows, but when it stands last it denotes a kind of unit. This distinction, however, is not always observed, since a change from one interpretation to another is frequently made in the same problem; but since both interpretations lead to the same result, no confusion ensues. For brevity and convenience of printing the symbol $\sqrt{-1}$ is replaced by the letter i, which will frequently be used with that meaning throughout the remainder of this chapter.

166. The meaning attached to $a\sqrt{-1}$, or ai, as a symbol of *quantity* determines the meaning to be attached to it as a symbol

of *operation.* To multiply a quantity is to substitute it for the unit in the multiplier. Now, ai indicates that a unit in length has been repeated a times and the result rotated through a right angle; therefore when these operations have been performed upon any quantity denoting a line, we say it has been multiplied by ai. It is important to observe that the *order* in which these two operations are performed does not affect the result.

Thus
$$b \times ai = ab \times i = bi \times a = abi = iab,$$
$$bi \times ai = abi \times i = b(i)^2 \times a = ab(i)^2 = -ab, \text{ etc.}$$

The product of any number of such factors depends only upon their numerical values and the number of rotations indicated by the operator i.

167. Since $ai \times ai = -a^2$, therefore $\sqrt{-a^2} = ai$; similarly, $\sqrt{-2} = \sqrt{2} . \sqrt{-1}$, etc., so that i is the only necessary symbol of imaginary expressions. But in this connection one point requires careful attention:

When a and b denote positive quantities, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$; but this is not true when a and b are negative. For example,

$$\sqrt{-4} \times \sqrt{-9} = 2\sqrt{-1} \times 3\sqrt{-1} = 6(\sqrt{-1})^2 = -6; \quad \text{Art. 166}$$

but the rule just quoted gives

$$\sqrt{-4} \times \sqrt{-9} = \sqrt{-4 \times -9} = \sqrt{36} = 6,$$

which is not true. The explanation is found in the fact that the formula $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ asserts that the operations of multiplication and the extraction of the square root obey the law of commutation (Art. 3, 3), but $\sqrt{-4}$ *does not indicate the extraction of the square root of* -4 (Arts. 160 and 162); therefore there is no reason why these symbols should obey the specified law.

168. The meaning of division by an imaginary quantity is derived from that of multiplication in the usual way—by defining

the quotient to be that quantity which, when multiplied by the divisor, will give the dividend as product.

Thus, since $OA \times \sqrt{-1} = OB$, $\therefore\ OB \div \sqrt{-1} = OA$; Art. 163

that is, the effect of $\sqrt{-1}$ as a *divisor* is to turn a line *backwards, i.e.*, in the negative direction through a right angle. We see, therefore, that $\sqrt{-1}$ as a *divisor* is equivalent to $-\sqrt{-1}$ as a *multiplier*, or that $\sqrt{-1}$ and $-\sqrt{-1}$, which, as *quantities*, denote opposition in direction, as *operators*, denote the reciprocals of each other.

169. The following is a concise statement of the results thus far obtained:

1. An **Imaginary Unit**, as a *quantity*, denotes a line of unit length drawn at right angles to a given fixed line; as a *multiplier* it turns a line through a right angle. In either case it is denoted by $\sqrt{-1}$.

2. An **Imaginary Quantity** is a number of imaginary units taken either positively or negatively; it is denoted by either $\sqrt{-1}\,.\,a$ or $a\sqrt{-1}$, where a may have any numerical value, either positive or negative. The same symbols are also used to denote operations.

3. As a quantity, $\pm a\sqrt{-1}$ consists of three elements—the number a, which denotes *distance;* the symbol $\sqrt{-1}$, which denotes *perpendicularity;* and the sign $+$ or $-$, which distinguishes the two directions along this perpendicular.

4. Imaginary quantities can be added or subtracted in the same way as positive or negative quantities, since they are measured in opposite directions on the same straight line.

5. Imaginary quantities can be multiplied (or divided) by combining the product (or quotient) of the real factors with the factors, 1, $\sqrt{-1}$, -1, or $-\sqrt{-1}$, according as the number of

imaginary factors, when divided by 4, gives 0, 1, 2 or 3 for remainder. Imaginary units, as divisors, may (when necessary) be turned into the equivalent multipliers Art. 168.

6. Since imaginary numbers and real numbers denote distances from a fixed point along two lines at right angles to each other, an imaginary number can never be equivalent to a real number. If, therefore, $a + b\sqrt{-1} = 0$, then a and b must separately vanish.

COMPLEX NUMBERS.

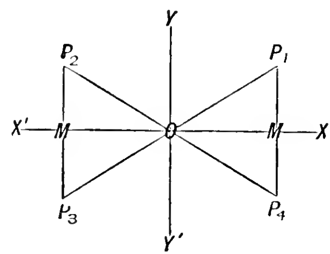
170. We have now assigned an intelligible meaning to imaginary quantities, and have shown that, with this meaning, two such quantities may be added or subtracted in the same way as real quantities. We have also assigned a meaning to the operations of multiplication and division of two quantities, providing any quantity is wholly real or wholly imaginary. It remains to determine what meaning should be attached to the sum of a real and an imaginary quantity, and to the operations of multiplication and division with such combinations, in order that the whole may be in harmony with the definitions and rules of Elementary Algebra, and with what has already been determined with regard to pure imaginaries.

171. A **Complex Number** is the sum of a real and an imaginary number. Its general form is $a + ib$, where a and b may have any numerical value, positive or negative, *i.e.*, a and b may be any quantities which do not involve the imaginary symbol i.

The exact meaning of the word "sum" should be noted, both when it refers to quantities (actual quantities, not to their representative symbols) and to algebraic expressions. The sum of two quantities is the quantity formed by combining the given quantities. The sum of two algebraical expressions is the combination of symbols which correctly represents the quantity formed by combining the quantities represented by the given expressions.

172. *To represent the sum of a real and an imaginary number, i.e., a complex number.*

Let $a+ib$ be the complex number, a and b being both positive. Take $X'OX$, YOY', two straight lines at right angles to each other. Let real numbers be measured from O in directions OX or OX', according as they are positive or negative; then imaginary numbers must be measured in directions OY or OY', according as the sign of the real factor is positive or negative. From O take OM in direction OX and a units in length; from M take MP_1 in direction OY and b units in length; then OM and MP_1 are correctly represented in magnitude and direction by $+a$ and $+ib$ respectively. Now, the result of a motion from O to M, followed by (or plus) a motion from M to P_1, is the same as a motion from O to P_1; therefore with this extension of the meaning of the sign $+$, OP_1 is the correct representative, both in magnitude and direction, of the complex number $a+ib$.



Similarly $OP_2 = -a+ib$, $OP_3 = -a-ib$, $OP_4 = a-ib$.

It is evident that the point P_1 might be reached by first measuring b units in direction OY, and then a units in direction OX. Therefore $a+ib = ib+a$.

173. The preceding Art. should be carefully compared with the addition of positive and negative numbers (Part I., Art. 34). To add a positive units and b negative units we measure a units in the positive direction, and from the extremity of this line measure b units in the negative direction. The *distance* and *direction* of the extremity of the latter line is taken for the sum of the two numbers; and this is precisely the method adopted in the preceding Art. In both cases the sum of the *lengths* of the two lines added is greater than the length of the line taken for their sum; but in both cases *direction* as well as length is con-

sidered in the addition, and it is this element which causes the difference.

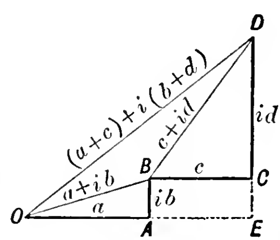
174. The **Modulus** of a complex number, $a+ib$, is the positive value of $\sqrt{a^2+b^2}$, which may be considered the absolute, or numerical, value of the expression. It will be observed that it represents the length of the line OP_1 without regard to direction; so that if a circle be described from O as centre, with radius equal to $\sqrt{a^2+b^2}$, an indefinite number of complex numbers may be represented, each of which has the same modulus, viz., the radius of the circle.

175. The **Argument** of a complex number is the angle through which the line of positive, real units must be rotated to correspond with the line denoting the complex number; its magnitude is determined by the signs of a and b, together with their *relative* numerical value. The four numbers, $a+ib$, $-a+ib$, $-a-ib$, $a-ib$, represented by OP_1, OP_2, OP_3, OP_4, have each the same modulus, but different arguments. Sometimes it is convenient to consider the argument negative. Thus the argument of $a-ib$ is the acute angle MOP_4 taken negatively; for a rotation through this angle in the *negative* direction gives the same result as a rotation through the corresponding reflex angle in the *positive* direction.

176. *To find the sum of two complex numbers.*

Let $a+ib$ and $c+id$ be the given numbers.

Draw the lines, OA, AB, BC, CD, representing the numbers, a, ib, c, id, in magnitude and direction, and join the various points as indicated in the figure.



Then $\quad OE=a+c,\quad ED=b+d,\quad$ Geometrically

and $\quad OB=a+ib,\quad BD=c+id.\quad$ Art. 172

Therefore $\quad OD=OE+i\,.\,ED=(a+c)+i(b+d).$

Now, with the extended meaning given to addition (Art. 173),

$$OB + BD = OD.$$

Therefore $\qquad (a + ib) + (c + id) = OB + BD$

$$= OD$$

$$= (a + c) + i(b + d),$$

from which we see that the sum of two complex numbers is found by adding the real and the imaginary parts separately.

177. A careful study of the figure in the preceding Art. will show the truth of the following statements:

1. The result of combining the four numbers, a, ib, c, id, is independent of the *order* in which they are taken, *i.e.*, the symbols obey the Commutative Law.

2. The numbers may be combined singly or in groups, *i.e.*, they obey the Distributive Law.

3. The meaning of Subtraction and the method of performing it follow at once from the meaning of Addition and the method of performing it. For example,

$$(a + ib) - (c + id) = (a - c) + i(b - d).$$

4. The modulus of the sum of two complex numbers cannot exceed the sum, nor be less than the difference, of their moduli, but may have any intermediate value. For example, the *length* of OD is less than the sum, but greater than the difference, of the lengths of OB and OD. But if the ratios $a : b$ and $c : d$ were equal, then OBD would be a straight line, and the modulus of the sum of the two numbers would equal the sum or difference of their moduli, according as the signs of a and b were the same as, or different from, those of c and d.

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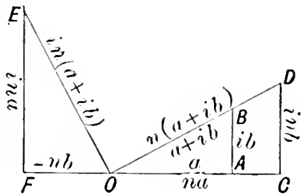
178. *To find the product of a complex number, (1) by a real number; (2) by an imaginary number.*

1. Let it be required to multiply the complex number $a+ib$ by the real number n.



Draw OB representing $a+ib$. Take $OD = n \, . \, OB$; then OD represents the result required. Through D draw DC parallel to BA, meeting OA produced in C; then from similar triangles, OAB, OCD, we have

$$OB : OA : AB = OD : OC : CD. \qquad \text{Euc. VI. 4}$$

But $\quad OD = n \, . \, OB, \;\therefore\; OC = n \, . \, OA$ and $CD = n \, . \, AB$.

Then
$$\begin{aligned} n(a+ib) &= n \, . \, OB = OD \\ &= OC + CD \qquad \text{Art. 172} \\ &= na + inb. \end{aligned}$$

2. Let it be required to multiply the complex number $a+ib$ by the imaginary number in.

Turn OD through a right angle into position OE; then OE represents $i \, . \, n(a+ib)$ (Art. 162). Draw EF at right angles to CO produced; then the triangles ODC and EOF are geometrically equal, and $CD = OF$ and $OC = FE$. But considering *direction* as well as length, $OF = -nb$ and $FE = ina$.

Therefore
$$\begin{aligned} in(a+ib) &= i \, . \, OD = OE \\ &= OF + FE \qquad \text{Art. 172} \\ &= -nb + ina \\ &= ina - nb. \end{aligned}$$

Thus both these operations obey the Distributive Law.

179. The meaning attached to $a+ib$ as a quantity, in connection with the definition of Multiplication, determines the meaning of $a+ib$ as a multiplier; for the quantity $a+ib$ is formed by adding two lines, the first of which is drawn in the direction

of the original unit and a times its length, and the second drawn at right angles to its extremity and b units in length. If, then, this operation be performed upon *any* line (since any line may be considered a unit), it is said to be multiplied by $a+ib$.

It will be observed that this operation turns the line multiplied through the angle indicated by the multiplier, and multiplies its length by the modulus.

180. *To find the product of two complex numbers.*

Let it be required to multiply $a+ib$ by $m+in$.

Draw the line representing $a+ib$; then from the meaning of multiplication by a complex number we have

$$\begin{aligned}(m+in)(a+ib) &= m(a+ib)+in(a+ib) && \text{Art. 179}\\ &= ma+imb+ina-nb && \text{Art. 178}\\ &= ma-nb+i(mb+na), && \text{Art. 176}\end{aligned}$$

which proves the Distributive Law when both numbers are complex. The student should draw the diagram corresponding to this operation, when it will be found that an independent geometrical proof may easily be given.

Similarly it may easily be shown that the Commutative Law is applicable in this and the preceding cases of multiplication.

181. *The modulus of the product of two complex numbers is equal to the product of their moduli, and the argument of the product is equal to the sum of their arguments.*

From the product given in the previous Art. we have

$$\begin{aligned}(ma-nb)^2+(mb+na)^2 &= m^2a^2-2mnab+n^2b^2\\ &\quad+m^2b^2-2mnab+n^2b^2\\ &=(m^2+n^2)(a^2+b^2),\end{aligned}$$

which proves the first part of the proposition. The second part is at once evident from the meaning assigned to multiplication by a complex number (Art. 179).

182. *The modulus of the quotient of two complex numbers is equal to the quotient of their moduli, and the argument of the quotient is equal to the difference of their arguments.*

The truth of this proposition follows from the preceding by observing that the product of the divisor and quotient must give the dividend.

183. Two complex numbers which differ only in the sign of the imaginary part are said to be **conjugate** to each other. Thus $a + ib$ and $a - ib$ are conjugate complex numbers.

The modulus of a complex number is evidently equal to that of its conjugate. Their angles are also equal, but they lie on opposite sides of the line on which the real units are reckoned.

The sum and the product of two conjugate complex numbers are each real.

For $$(a + ib) + (a - ib) = 2a,$$
and $$(a + ib) \times (a - ib) = a^2 - i^2b^2$$
$$= a^2 + b^2.$$

The reader should illustrate these operations by a diagram.

184. A complex number vanishes when the real and the imaginary parts separately vanish; and conversely, if a complex number vanishes, the real and the imaginary parts must separately vanish. Both statements are at once evident from the diagram representing a complex number. They are also evident from the symbols, since if $a = 0$ and $b = 0$, then $a + ib = 0$. And if $a + ib = 0$, then $a = -ib$, a real number equal to an imaginary number, which is impossible (Art. 169, 6).

185. *If two complex numbers are equal, their real and their imaginary parts are separately equal.*

For if $$a + ib = c + id,$$
then $$a - c + i(b - d) = 0,$$

which is impossible unless $a - c$ and $b - d$ are each zero (Art. 169, 6).

This proposition simply asserts that the measurements (OM and MP_1) in directions OX and OY necessary to reach a fixed point (P_1) are themselves fixed and unchangeable.

186. *If, in any rational integral function of x, we substitute $a+ib$ for x and reduce the result to the form $P+iQ$, then $P-iQ$ is the result of substituting $a-ib$.*

For, since P is real, it can involve only *even* powers of ib; and since iQ is imaginary, it can involve only *odd* powers of ib. Therefore if the sign of ib be changed, iQ will simply change its sign, but P will remain unchanged.

Cor.—If $P=0$ and $Q=0$, then $x-(a+ib)$ is a factor of the given function, and consequently $x-(a-ib)$ is also a factor.

187. It will now be instructive to briefly review the course of reasoning already given in connection with imaginary quantities. The meaning first assigned to the symbol $\sqrt{-1}$ made it a symbol of *operation*, that of turning the direction of a line through a right angle; then in connection with a numerical factor we made it a symbol of *quantity*, denoting the quantity resulting from its use as an operator. Having fixed its meaning, both as a symbol of operation and a symbol of quantity, we examined the results of combining the quantities it represents with those represented by other symbols, and traced the connection between the operations performed on the quantities themselves and the symbolical operations by which they might conveniently be represented.

The various steps of this course of reasoning are the same as those by which the operations of Elementary Algebra were established; and since imaginaries, both when taken alone and when combined with other quantities, have been shown to obey the fundamental laws of algebraic operations, the whole forms one harmonious system, and results obtained by the use of imaginaries are quite as reliable as those obtained by any other process of mathematical investigation.

One point of interest still remains. The imaginary symbols, both of quantity and operation, are unintelligible except in connection with geometrical magnitudes. Suppose that in the solution of a problem relating to other magnitudes the imaginary symbols are used, but that they do not appear in the result, is the result reliable? To answer this we have only to observe that magnitudes of any kind may be represented by straight lines, and that by so doing the problem immediately becomes a geometrical one, and then all operations are intelligible. The result, when correctly interpreted, is therefore in such cases perfectly reliable.

188. We shall now investigate the properties of certain imaginary quantities which are frequently employed in mathematical investigations.

Suppose $$x = \sqrt[3]{1},$$

then $$x^3 = 1 \text{ or } x^3 - 1 = 0,$$

that is, $$(x-1)(x^2+x+1) = 0.$$

Therefore, either $$x - 1 = 0 \text{ or } x^2 + x + 1 = 0;$$

whence $$x = 1 \text{ or } x = \frac{-1 \pm \sqrt{-3}}{2}.$$

Each of these values of x when cubed gives unity; therefore unity has three cube roots, namely,

$$1, \quad \frac{-1+\sqrt{-3}}{2}, \quad \frac{-1-\sqrt{-3}}{2},$$

two of which are imaginary expressions. Denote these by p and q; then, since p and q are the roots of the equation,

$$x^2 + x + 1 = 0,$$

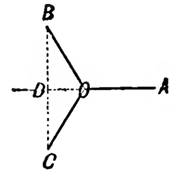
their product is equal to unity.

That is, $$pq = 1, \quad \therefore \ p^3 q = p^2.$$

But $$p^3 = 1, \quad \therefore \quad q = p^2.$$

Similarly we may show that $p = q^2$.

189. The geometrical meaning of these results will be found interesting and instructive.



Draw three lines, OA, OB, OC, each a unit in length, making angles of 120 degrees between each pair. Join BC; then it may easily be shown that BC cuts AO produced at right angles, and that $OD = \frac{1}{2}$ and $DB = \frac{1}{2}\sqrt{3}$. We have, then,

$$OB = -\frac{1}{2} + \frac{i\sqrt{3}}{2} = p,$$

$$OC = -\frac{1}{2} - \frac{i\sqrt{3}}{2} = q.$$

Now, since the absolute value of each of these expressions (the lengths of OB and OC) is unity, the absolute value of their product, or of any power of one of them, is unity; and since the sum of the angles AOB and (the reflex angle) AOC is 360 degrees, we see that their product is represented by OA, that is, $+1$. Again, $p^2 = q$, because turning *twice* through an angle of 120 degrees gives an angle of 240 degrees; and $q^2 = p$, because turning twice through an angle of 240 degrees gives a whole revolution and 120 degrees besides.

190. Since each of the imaginary roots is the square of the others, it is usual to denote the cube roots of unity by 1, ω, ω^2, where ω is either of the imaginary roots. The following are important properties of these quantities:

1. The sum of the three cube roots of unity is zero. For ω is a root of the equation,

$$x^2 + x + 1 = 0;$$

$$\therefore\ \omega^2 + \omega + 1 = 0.$$

2. Any three successive integral powers of ω give the three cube roots of unity (zero and negative powers included). For

any number n, when divided by 3, must give 0, 1 or 2 for remainder. Let m be the quotient; then

if $\quad n = 3m, \quad \omega^n = \omega^{3m} \quad = (\omega^3)^m \quad = 1;$

if $\quad n = 3m + 1, \quad \omega^n = \omega^{3m+1} = \omega^{3m} \cdot \omega \;= \omega;$

if $\quad n = 3m + 2, \quad \omega^n = \omega^{3m+2} = \omega^{3m} \cdot \omega^2 = \omega^2.$

3. Every number has three cube roots. Let a denote the cube root of a number found in the ordinary way; then $a\omega$ and $a\omega^2$ are also cube roots.

For
$$(a\omega)^3 = a^3\omega^3 = a^3,$$
$$(a\omega^2)^3 = a^3\omega^6 = a^3.$$

It will be observed that two of the cube roots are imaginary.

191. We shall now give a few examples:

Ex. 1.—Divide $c + di$ by $a + bi$.

$$\frac{c+di}{a+bi} = \frac{(c+di)(a-bi)}{(a+bi)(a-bi)}$$
$$= \frac{ac+bd+(ac-bd)i}{a^2+b^2}$$
$$= \frac{ac+bd}{a^2+b^2} + \frac{ac-bd}{a^2+b^2}i.$$

Thus by reference to Arts. 176, 177 and 180 we see that the *sum*, *difference*, *product* and *quotient* of two complex numbers are, in general, complex numbers. In special cases, however, the result may be either real or a pure imaginary.

Ex. 2.—To find the square root of $a + b\sqrt{-1}$.

Assume
$$\sqrt{a+b\sqrt{-1}} = x + y\sqrt{-1}$$
where x and y are real quantities.

Squaring,
$$a + b\sqrt{-1} = x^2 - y^2 + 2xy\sqrt{-1}.$$

Equating the real and also the imaginary parts

we get $$x^2 - y^2 = a \qquad (1)$$

and $$2xy = b. \qquad (2)$$

Therefore $$(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$$
$$= a^2 + b^2,$$

or $$x^2 + y^2 = \sqrt{a^2 + b^2}. \qquad (3)$$

From (1) and (3) we obtain

$$x^2 = \frac{\sqrt{a^2+b^2} + a}{2}, \qquad y^2 = \frac{\sqrt{a^2+b^2} - a}{2}.$$

Therefore $$x = \pm\left\{\frac{\sqrt{a^2+b^2} + a}{2}\right\}^{\frac{1}{2}}, \quad y = \pm\left\{\frac{\sqrt{a^2+b^2} - a}{2}\right\}^{\frac{1}{2}},$$

from which the required root is known.

In this example observe:

(1) In (3) the positive sign must be taken with the radical, because x and y being real, $x^2 + y^2$ is positive.

(2) The signs of x and y must be alike or different according as b is positive or negative, since $2xy = b$.

Ex. 3.—Find the square root of $\pm\sqrt{-1}$.

Assume $$\sqrt{\pm\sqrt{-1}} = x \pm y\sqrt{-1}.$$

Then $$\pm\sqrt{-1} = x^2 - y^2 \pm 2xy\sqrt{-1},$$

therefore $$x^2 - y^2 = 0 \text{ and } 2xy = 1,$$

from which $$x = y = \pm\frac{1}{\sqrt{2}}.$$

Therefore $$\sqrt{+\sqrt{-1}} = \pm\frac{1}{\sqrt{2}}(1 + \sqrt{-1}),$$

and $$\sqrt{-\sqrt{-1}} = \pm\frac{1}{\sqrt{2}}(1 - \sqrt{-1}).$$

The student should draw the diagram corresponding to these operations.

Ex. 4.—Resolve x^3+y^3 into three factors.

We have $\quad x^3+y^3=(x+y)(x^2-xy+y^2)$.

Now $\quad \omega+\omega^2=-1$ and $\omega \, . \, \omega^2=1$, Art. 190

therefore $\quad x^3+y^3=(x+y)(x+\omega y)(x+\omega^2 y)$.

Similarly $\quad x^3-y^3=(x-y)(x-\omega y)(x-\omega^2 y)$.

Ex. 5.—Resolve $x^2+y^2+z^2-xy-yz-zx$ into factors.

The expression may be written

$$x^2-(y+z)x+(y^2-yz+z^2).$$

We have now to find two quantities whose sum is $-(y+z)$ and whose product is y^2-yz+z^2. Factoring this latter expression we get $y+\omega z$ and $y+\omega^2 z$; but the sum of these two expressions is not $-(y+z)$. If we multiply the first factor by ω and the second by ω^2, their product will be unchanged, and the factors become $\omega y+\omega^2 z$ and $\omega^2 y+\omega z$, whose sum is $-(y+z)$ as required. Therefore

$$x^2+y^2+z^2-xy-yz-zx=(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z).$$

Ex. 6.—Factor $x^3+y^3+z^3-3xyz$.

The expression may be written in either of the forms,

$$x^3+(\omega y)^3+(\omega^2 z)^3-3x \, . \, \omega y \, . \, \omega^2 z, \quad x^3+(\omega^2 y)^3+(\omega z)^3-3x \, . \, \omega^2 y \, . \, \omega z.$$

From the original form we know that $x+y+z$ is a factor, therefore from the above forms we know that $x+\omega y+\omega^2 z$ and $x+\omega^2 y+\omega z$ are factors; and since the expression is of but three dimensions there can be no other literal factor. The coefficient of x^3 in the product of the three factors is the same as that of x^3 in the given expression; therefore

$$x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z).$$

The factors of the expression might evidently have been taken from those of the last example, and conversely. The methods of this example might also have been used in Ex. 4.

192. The application of imaginaries to the solution of geometrical problems does not fall within the scope of the present work. We give, however, a single example, to show the new power gained by the introduction of the imaginary symbol.

Ex.—Squares are described upon the three sides of any triangle, and their outer corners are joined. Show that the sum of the squares on these three lines equals three times the sum of the squares on the sides of the triangle.

Upon BC, CA, AB, the sides of the triangle, describe squares $BEFC$, $CGHA$, $AKLB$; draw AD perpendicular to BC, and denote the *lengths* of BC, CA, AB, BD, AD by a, b, c, x, y. Then, considering *direction* as well as length,

$$BA = BD + DA, \qquad CA = CD + DA,$$

or
$$c = x + iy, \qquad b = -(a - x) + iy.$$

Now, the area of a square is the square of the *length* of the side, *i.e.*, it is the *square of the modulus* when the side is expressed by a complex number; therefore

$$\begin{aligned} a^2 + b^2 + c^2 &= a^2 + \{(a-x)^2 + y^2\} + (x^2 + y^2) \\ &= 2(a^2 - ax + x^2 + y^2). \end{aligned}$$

Again,
$$\begin{aligned} FG &= FC + CG & EL &= EB + BL \\ &= -i\,.\,CB - i\,.\,CA & &= i\,.\,BC + i\,.\,BA \\ &= ia - i(x - a + iy) & &= ia + i(x + iy) \\ &= y + i(2a - x), & &= -y + i(x + a), \end{aligned}$$

and $KH = KA + AH = -i(x + iy) - i(x - a + iy) = 2y - i(2x - a)$.

Then, as before, sum of squares on FG, EL, KH

$$\begin{aligned} &= \{y^2 + (2a - x)^2\} + \{y^2 + (x + a)^2\} + \{4y^2 + (2x - a)^2\} \\ &= 6(a^2 - ax + x^2 + y^2), \end{aligned}$$

which proves the proposition.

NOTE.—In the above draw BC horizontally *to the right*, DA *upwards*; choose the usual directions for measurement and rotation, and carefully distinguish between *positive* and *negative* throughout.

EXERCISE XVI.

1. Find the values of $x^8+x^6+x^4+x^2+1$ and $x^7+x^5+x^3+x$ when $x=2i$.

2. Simplify $(2-3i)(3-2i)+(1-i\sqrt{3})^2 i$.

3. Express as complex numbers, $(2-3i)^3$ and $(1+i)^5$.

4. Simplify

$(2-i\sqrt{5})^3+(2+i\sqrt{5})^3$ and $\{(2+i\sqrt{5})^2-(2+i\sqrt{3})^2\}(\sqrt{5}+\sqrt{3})$.

5. Simplify $\dfrac{3-i}{2-3i}+\dfrac{1-3i}{3-2i}$ and $\dfrac{3-i\sqrt{2}}{i\sqrt{3}-\sqrt{2}}+\dfrac{8i+\sqrt{2}}{\sqrt{3}+i\sqrt{2}}$.

6. Extract the square root of $5-12i$, $1-4i\sqrt{3}$ and $4i\sqrt{5}-1$.

7. Extract the cube root of $-3\sqrt{3}-7i\sqrt{2}$ and $-10+9i\sqrt{3}$.

8. Find the values of $a^3+b^3+c^3-3abc$ when

$$a=i,\quad b=i+\sqrt{3},\quad c=i-\sqrt{3}.$$

9. Show by the use of imaginaries that $(a^3-3ab^2)^2+(3a^2b-b^3)^2=(a^2+b^2)^3$, and deduce similar expressions for other powers of a^2+b^2.

10. Simplify

$$(x-1-i\sqrt{2})(x-1+i\sqrt{2})(x-2-i\sqrt{3})(x-2+i\sqrt{3}).$$

11. Divide $14-\sqrt{15}-(7\sqrt{3}+2\sqrt{5})i$ by $7-i\sqrt{5}$.

12. Simplify $\dfrac{7+i\sqrt{3}}{2-i\sqrt{3}}+\dfrac{8+3i\sqrt{3}}{2+i\sqrt{3}}-\dfrac{4(2-i\sqrt{3})}{1-i\sqrt{3}}$.

13. Find the modulus of $3+4i$, m^2-n^2+2mni and $\dfrac{1+i}{1-i}$.

14. Express $\dfrac{69-7\sqrt{15}+(\sqrt{3}-6\sqrt{5})i}{3-(\sqrt{3}-3\sqrt{5})i}$ in the form $a+ib$.

15. Find the value of $\frac{a^3}{b^3}\left(\frac{1}{a^3}-\frac{1}{b^3}\right)-\frac{b^3}{a^3}\left(\frac{1}{a^3-b^3}-\frac{1}{a^3+b^3}\right)$ when $a^2+b^2=0$ and $ab=1$.

16. Find the modulus of $\frac{(2-3i)(3+4i)}{(6+4i)(15-8i)}$.

17. Find the product of $(a+ib)(ia+b)(ia+ib)$.

18. Express $(a+ib)(b+ic)(c+ia)$ as a complex number.

19. Show that

$$\{(2a-b-c)+i(b-c)\sqrt{3}\}^3=\{(2b-c-a)+i(c-a)\sqrt{3}\}^3$$

What relation exists between these quantities *before* they are cubed?

20. If $p+qi$ is a root of $ax^2+bx+c=0$, then $ap^2+bp+c=aq^2$ and $2ap+b=0$; a, b and c denoting real quantities. Show the necessity for the latter clause in this example.

21. Find the sum of $1+2i+3i^2+\ldots(n+1)i^n$ when n is (1) an even multiple of 2; (2) an odd multiple of 2.

22. Find the product of $(a+b-ci)(b+c-ai)(c+a-bi)$.

23. Detect the fallacy in the following reasoning:

$$(-1)^{\frac{1}{2}}=(-1)^{\frac{2}{4}}=\{(-1)^2\}^{\frac{1}{4}}=(+1)^{\frac{1}{4}}=1,$$

and illustrate by reference to a geometrical diagram.

24. Find the modulus of $1+ix+i^2x^2+\ldots$ *ad inf.*, where $x<1$.

25. If ω is an imaginary cube root of unity, then $1+\omega$ and $1+\omega^2$ are the imaginary cube roots of -1.

26. Show that $(1+\omega)^2$ and $(2+\omega)^2$ are cube roots of 1 and -27, and find the other roots.

27. Find the values of

$$(1+\omega)^4+(1+\omega^2)^4 \text{ and } (1-\omega+\omega^2)(1+\omega-\omega^2).$$

28. Simplify

$$(\omega+i)(\omega^2-i) \text{ and } (1+\omega-\omega^2)^2+(1-\omega+\omega^2)^2+(1-\omega-\omega^2)^2.$$

29. Show that $(1-\omega+\omega^2)^{3n}=(1+\omega-\omega^2)^{3n}=(-8)^n$.

30. Express $\dfrac{1-\omega}{1+\omega}$, $\dfrac{1+\omega}{1-\omega}$ and $\dfrac{\omega+i}{\omega-i}$ with rational denominators.

31. Show that

$$(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)\ldots\text{ to } 2n \text{ factors}=2^{2n}.$$

32. Show that $\dfrac{\omega}{i}$, $\dfrac{i}{\omega}$ and $i\omega$ are sixth roots of -1. Find the other three, and illustrate geometrically.

33. Show that

$$\frac{1+i}{\sqrt{2}}\div\frac{\sqrt{3}+i}{2}=\frac{1+i\sqrt{3}}{2}\div\frac{1+i}{\sqrt{2}}=\left(\frac{\sqrt{3}+i}{2}\right)^{\frac{1}{2}}=\left(\frac{1+i}{\sqrt{2}}\cdot\frac{\sqrt{3}+i}{2}\right)^{5},$$

and give the geometrical meaning of these equalities.

34. Simplify $\left(x-\dfrac{\omega}{i}\right)\left(x-\dfrac{i}{\omega}\right)(x-i\omega)\left(x-\dfrac{1}{i\omega}\right)(x-i)\left(x-\dfrac{1}{i}\right)$.

Show that the result will be unchanged by changing the connecting signs of each of the factors, or by multiplying each of the second terms by ω; but if each be multiplied by i, the connecting sign of the result will be changed. Give the geometrical meaning of each of these statements.

35. Simplify $\dfrac{1}{a+b+c}+\dfrac{1}{a+b\omega+c\omega^2}+\dfrac{1}{a+b\omega^2+c\omega}$.

36. If $x+y+z=\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=0$ and $xyz=1$, show that x, y, z are the cube roots of unity.

37. If $x+y+z=\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=0$, show that $x^8+y^8+z^8=0$ and that $x^9+y^9+z^9=xyz(x^6+y^6+z^6)$.

38. If $x=a+b$, $y=a\omega+b\omega^2$, $z=a\omega^2+b\omega$, show that

(1) $x^2+y^2+z^2=6ab$, (2) $x^3+y^3+z^3=3(a^3+b^3)$,

(3) $x^4+y^4+z^4=18a^2b^2$, (4) $x^5+y^5+z^5=15ab(a^3+b^3)$,

(5) $xyz=a^3+b^3=-(x+y)(y+z)(z+x)$.

39. If $X = ax + cy + bz,\quad Y = cx + by + az,\quad Z = bx + ay + cz,$
show that $\quad X + \omega Y + \omega^2 Z = (a + \omega c + \omega^2 b)(x + \omega^2 y + \omega z)$
and that $\quad X^2 + Y^2 + Z^2 - YZ - ZX - XY$

$$= (a^2 + b^2 + c^2 - bc - ca - ab)(x^2 + y^2 + z^2 - yz - zx - xy).$$

40. If $x + y + z = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 0$, or if $a + b + c = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = 0$,

then $(ax + by + cz)^6 + (bx + cy + az)^6 + (cx + ay + bz)^6$

$$= 3\{(ax + bu + cz)(bx + cy + az)(cx + ay + bz)\}^2.$$

CHAPTER XII.

QUADRATIC AND HIGHER EQUATIONS.

ONE UNKNOWN QUANTITY.

193. The symbol denoting the unknown quantity in an equation is frequently called a **Variable.** Symbols denoting other quantities are called **Constants.**

It is often necessary to examine the result of assigning special values to one or more letters in an algebraical expression; the letters to which different values are thus given are also called variables.

194. An **Integral Equation** is one in which the variable or unknown quantity does not appear in the denominator of a fraction, and is not affected by any root sign. It is in its simplest form when its terms are arranged in powers of the variable, and the coefficient of the highest power is unity and positive.

195. The **Degree of an Equation** is the number of dimensions in the highest power of the variable which occurs in the equation. The words, "linear," "quadratic," "cubic," "biquadratic," are used to denote equations of the first, second, third and fourth degrees respectively. These terms are especially applied to integral equations.

196. From an equation which is irrational, or fractional, or both, an integral equation can be derived, the roots of which are usually assumed to be the roots of the original equation. Upon

trial, however, it is frequently found that one or more roots of one equation will not apparently satisfy the others. This point deserves the most careful consideration.

197. The following method of rationalizing surd equations is instructive. Arrange all the terms on one side and denote them by a, b, c, etc., the rational quantities, if any, being collected into one term.

(1) Let there be two terms.

Then the equation is $$a+b=0.$$
Therefore $$(a-b)(a+b)=0,$$
or $$a^2-b^2=0,$$
which will be rational, since each term is a square.

(2) Let there be three terms.

Then the equation is $$a+b+c=0.$$
Therefore $$(a+b-c)(a-b+c)(a-b-c)(a+b+c)=0,$$
or $$a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2=0,$$
which will be rational, since each term is raised to an *even* power.

This method may easily be extended to four or more terms, the results in each case being the same as would be obtained by the ordinary method of squaring.

198. In Part I., Exercises XC., XCI. and XCII., illustrations were given of ordinary quadratic equations of one unknown. We now proceed to give specimens of more difficult quadratics, and of equations of higher degree, that can be solved either as quadratics or by other artifices. It is well, however, to bear in mind that no text-book can afford space for illustrations of all the different ingenious artifices employed to obtain solutions of abstruse and complicated equations.

Ex. 1.—Solve $\left(\frac{x}{x-1}\right)^2 + \left(\frac{x}{x+1}\right)^2 = n(n-1).$ (1)

Let $$n(n-1) = m.$$

Then $$\left(\frac{x}{x-1}\right)^2 + \left(\frac{x}{x+1}\right)^2 = m. \qquad (2)$$

Clearing of fractions and collecting coefficients,

$$x^4(2-m) + x^2(2+2m) = m,$$

or $$x^4 + \frac{x^2(2+2m)}{2-m} = \frac{m}{2-m}.$$

Completing the square on left-hand side of equation,

$$\left(x^2 + \frac{1+m}{2-m}\right)^2 = \frac{4m+1}{(2-m)^2}.$$

Substituting $n(n-1)$ for m,

$$\left(x^2 + \frac{n^2-n+1}{2-n^2+n}\right)^2 = \frac{4n^2-4n+1}{(2-n^2+n)^2}.$$

Extracting square root,

$$x^2 + \frac{n^2-n+1}{2-n^2+n} = \pm\frac{2n-1}{2-n^2+n}.$$

$$\therefore\ x^2 = \frac{n^2-3n+2}{n^2-n-2} \text{ or } \frac{n^2+n}{n^2-n-2}$$

$$= \frac{n-1}{n+1} \text{ or } \frac{n}{n-2}.$$

N.B.—Instead of completing the square to obtain the root, factoring might have been used.

Ex. 2.—Solve $2^{x+1} + 4^x = 80$.

$$2^{x+1} + 4^x = 2 \,.\, 2^x + 2^{2x} = 80,$$

or $$2^{2x} + 2 \times 2^x = 80.$$

Let $$y = 2^x.$$

Then $$y^2 + 2y = 80, \text{ or } y^2 + 2y - 80 = 0.$$

Factoring, $$(y + 10)(y - 8) = 0.$$

Therefore $$y = -10 \text{ or } 8,$$

that is, $$2^x = -10 \text{ or } 8.$$

Let $$2^x = 8 = 2^3.$$

$$\therefore\ x = 3.$$

If we let $$2^x = -10,$$

the solution cannot be obtained.

Ex. 3.—Solve $(x + a)(x + 2a)(x + 3a)(x + 4a) = c^4$.

Multiply together first and fourth factors, and second and third factors, since the sum of a and $4a$ is the same as the sum of $2a$ and $3a$.

Then $$(x^2 + 5ax + 4a^2)(x^2 + 5ax + 6a^2) = c^4.$$

Let $$x^2 + 5ax + 4a^2 = y.$$

Then $$y(y + 2a^2) = c^4,$$

or $$y^2 + 2a^2y = c^4.$$

Completing square, $$(y + a^2)^2 = c^4 + a^4.$$

Extracting root, $$y + a^2 = \pm \sqrt{c^4 + a^4},$$

or $$y = -a^2 \pm \sqrt{c^4 + a^4}.$$

If we now substitute back again for y its value $x^2 + 5ax + 4a^2$, there will remain to be solved the ordinary quadratic,

$$x^2 + 5ax + 4a^2 = -a^2 \pm \sqrt{c^4 + a^4},$$

or $$x^2 + 5ax = -5a^2 \pm \sqrt{c^4 + a^4}.$$

This equation, although cumbrous, offers no difficulty in its solution.

Ex. 4.—Solve $x^2 + x^{-2} + x + x^{-1} = 4$.

Add 2 to each side of the equation.

Then $$(x^2 + 2 + x^{-2}) + (x + x^{-1}) = 6,$$

or $$(x + x^{-1})^2 + (x + x^{-1}) - 6 = 0,$$

or $$(x + x^{-1} + 3)(x + x^{-1} - 2) = 0.$$

Therefore $$x + x^{-1} + 3 = 0, \qquad (1)$$

or $$x + x^{-1} - 2 = 0. \qquad (2)$$

Solving in turn $x + x^{-1} + 3 = 0$ and $x + x^{-1} - 2 = 0$ we find

$$x = -\frac{3 \pm \sqrt{5}}{2}, \quad x = 1,\ 1.$$

Ex. 5.—Solve $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0$.

Arrange as follows:

$$(x^5 + 1) + 2x(x^3 + 1) - 3x^2(x + 1) = 0. \qquad (1)$$

It is evident that $x + 1$ is a factor of each quantity in brackets, $\therefore\ x = -1$ is one solution.

Dividing (1) by $(x + 1)$ we obtain

$$(x^4 - x^3 + x^2 - x + 1) + 2x(x^2 - x + 1) - 3x^2 = 0,$$

or $$x^4 + x^3 - 4x^2 + x + 1 = 0,$$

or $$(x^4 + 1) + x(x^2 + 1) = 4x^2.$$

Adding $2x^2$ to each side of the equation,

$$(x^4 + 2x^2 + 1) + x(x^2 + 1) = 6x^2,$$

or $$(x^2 + 1)^2 + x(x^2 + 1) + \frac{x^2}{4} = \frac{25x^2}{4}.$$

Extracting square root,

$$x^2 + 1 + \frac{x}{2} = \pm\frac{5x}{2}.$$

We have now two quadratics to be solved, viz.:

$$x^2+1+\frac{x}{2}=-\frac{5x}{2},$$

and

$$x^2+1+\frac{x}{2}=+\frac{5x}{2},$$

neither of which presents any difficulty.

Exs. 4 and 5 are specimens of *reciprocal* equations, *i.e.*, equations whose coefficients, equidistant from beginning and end, are the same when the terms are arranged according to the powers of x.

Reciprocal equations may also be defined as those which are not altered by changing x into $\frac{1}{x}$.

Every reciprocal equation of *odd* degree will be divisible by $x-1$ or $x+1$ (see Theory of Divisors), according as the last term is -1 or $+1$; and every reciprocal equation of *even* degree with its last term -1 will be divisible by x^2-1; and the reduced equation after the division will be found to be reciprocal, of an even degree, and with its last term $+1$ (Colenso's Algebra).

Thus a *cubic* (reciprocal) may be reduced to a *quadratic*, and one of the *fifth* or *sixth* degree to a *biquadratic*, and then solved as in Ex. 4.

Ex. 6.—Solve $\dfrac{1+x^4}{(1+x)^4}=a.$

Add 1 to each side.

Then

$$\frac{(1+x)^4+1+x^4}{(1+x)^4}=1+a,$$

or

$$\frac{2(1+x+x^2)^2}{(1+x)^4}=1+a,$$

or

$$\frac{(1+x+x^2)^2}{(1+x)^4}=\frac{1+a}{2}.$$

Extracting square root,

$$\frac{1+x+x^2}{(1+x)^2} = \pm\frac{\sqrt{2(1+a)}}{2},$$

or
$$\frac{1+x+x^2}{1+2x+x^2} = \pm\frac{\sqrt{2(1+a)}}{2} = m,$$

or
$$1 - \frac{x}{1+2x+x^2} = m,$$

or
$$\frac{1+2x+x^2}{x} = \frac{1}{1-m}.$$

This now can be solved as a *quadratic.*

Ex. 7.—Solve $\sqrt[m]{(1+x)^2} - \sqrt[m]{(1-x)^2} = \sqrt[m]{1-x^2}$,

that is,
$$(1+x)^{\frac{2}{m}} - (1-x)^{\frac{2}{m}} = (1-x^2)^{\frac{1}{m}}.$$

Dividing by $(1-x)^{\frac{2}{m}}$,
$$\left(\frac{1+x}{1-x}\right)^{\frac{2}{m}} - 1 = \left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}.$$

Assume
$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} = y.$$

Then
$$y^2 - 1 = y,$$

or
$$y^2 - y = 1.$$

Solving for y we get
$$y = \frac{1}{2} \pm \frac{\sqrt{5}}{2},$$

that is,
$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} = \frac{1 \pm \sqrt{5}}{2},$$

$$\therefore \frac{1+x}{1-x} = \left(\frac{1 \pm \sqrt{5}}{2}\right)^m$$

$$= \frac{(1 \pm \sqrt{5})^m}{2^m}.$$

Adding numerators and denominators, and dividing by their difference,

$$x=\frac{(1\pm\sqrt{5})^m-2^m}{(1\pm\sqrt{5})^m+2^m}.$$

Ex. 8.—Solve $\left(\frac{2x+3}{2x-3}\right)^{\frac{1}{3}}+\left(\frac{2x-3}{2x+3}\right)^{\frac{1}{3}}=\frac{8}{13}\left(\frac{4x^2+9}{4x^2-9}\right).$

Now, $$\frac{2x+3}{2x-3}+\frac{2x-3}{2x+3}=\frac{2(4x^2+9)}{4x^2-9},$$

$$\therefore\quad \left(\frac{2x+3}{2x-3}\right)^{\frac{1}{3}}+\left(\frac{2x-3}{2x+3}\right)^{\frac{1}{3}}=\frac{4}{13}\left\{\frac{2x+3}{2x-3}+\frac{2x-3}{2x+3}\right\}.$$

Assume $$\left(\frac{2x+3}{2x-3}\right)^{\frac{1}{3}}=y.$$

Then $$\left(y+\frac{1}{y}\right)=\frac{4}{13}\left(y^3+\frac{1}{y^3}\right).$$

It is evident that $y+\frac{1}{y}$ is a factor of both sides, and therefore $y+\frac{1}{y}=0$ will give a partial solution.

If $$y+\frac{1}{y}=0,\ \text{then}\ y^2=-1,$$

or $$\left(\frac{2x+3}{2x-3}\right)^{\frac{2}{3}}=-1.$$

Cubing, $$\left(\frac{2x+3}{2x-3}\right)^2=-1,$$

or $$x=\pm\frac{3}{2}\sqrt{-1}.$$

To obtain the remaining solutions we have

$$1=\frac{4}{13}\left(y^2-1+\frac{1}{y^2}\right),$$

or $$y^2+\frac{1}{y^2}=\frac{17}{4}.$$

Solving, $$y = 2, \ \frac{1}{2}, \ -2, \ -\frac{1}{2},$$

that is, $$\left(\frac{2x+3}{2x-3}\right)^{\frac{1}{3}} = 2, \text{ or } \frac{1}{2}, \text{ or } -1, \text{ or } -\frac{1}{2},$$

or $$\frac{2x+3}{2x-3} = 8, \text{ or } \frac{1}{8}, \text{ or } -8, \text{ or } -\frac{1}{8}.$$

$$\therefore \ x = \pm\frac{27}{14}, \ \pm\frac{7}{6}.$$

Ex. 9.—Solve $\sqrt{x^2+ax-1} + \sqrt{x^2+bx-1} = \sqrt{a} + \sqrt{b}$. (1)

Now, $$(x^2+ax-1) - (x^2+bx-1) = x(a-b),$$

that is, $$(\sqrt{x^2+ax-1})^2 - (\sqrt{x^2+bx-1})^2 = x(a-b), \qquad (2)$$

Dividing (2) by (1),
$$\sqrt{x^2+ax-1} - \sqrt{x^2+bx-1} = x(\sqrt{a} - \sqrt{b}) \qquad (3)$$

Adding (1) and (3) we obtain
$$2\sqrt{x^2+ax-1} = \sqrt{a} + \sqrt{b} + x(\sqrt{a} - \sqrt{b}).$$

Let $$\sqrt{a} + \sqrt{b} = m \text{ and } \sqrt{a} - \sqrt{b} = n.$$

$$\therefore \ 2\sqrt{x^2+ax-1} = m + nx.$$

Squaring, $$4x^2 + 4ax - 4 = m^2 + 2mnx + n^2x^2.$$

Transposing, and arranging according to powers of x,
$$x^2(4-n^2) + x(4a-2mn) - (4+m^2) = 0.$$

But $$mn = (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b,$$

$$\therefore \ x^2(4-n^2) + x(4a-2a+2b) - (m^2+4) = 0,$$

or $$x^2(4-n^2) + x(2a+2b) - (m^2+4) = 0.$$

Factoring, $$\{x(4-n^2) + (m^2+4)\}(x-1) = 0 \qquad (4)$$

$\{$since $m^2 + 4 + n^2 - 4 = m^2 + n^2$
$$= (\sqrt{a} + \sqrt{b})^2 + (\sqrt{a} - \sqrt{b})^2 = 2a + 2b\}.$$

From (4) we obtain $x = 1$ or $\dfrac{m^2+4}{n^2-4}$,

that is, $x = 1$ or $\dfrac{(\sqrt{a}+\sqrt{b})^2+4}{(\sqrt{a}-\sqrt{b})^2-4}$.

Ex. 10.—Solve $\dfrac{(a-x)^4+(x-b)^4}{(a-x)^2+(x-b)^2} = \dfrac{41}{20}(a-b)^2$.

Assume $a - x = m$ and $x - b = n$.

$$\therefore\ m + n = a - b,$$

$$\therefore\ \frac{m^4+n^4}{m^2+n^2} = \frac{41}{20}(a-b)^2 = \frac{41}{20}(m+n)^2,$$

$$\therefore\ (m^4+n^4) = \frac{41}{20}(m^2+n^2)(m+n)^2,$$

or $$20(m^4+n^4) = 41(m^2+n^2)(m^2+n^2+2mn),$$

or $$21(m^4+n^4)+82m^2n^2+82mn(m^2+n^2) = 0.$$

Arranging according to powers of m,

$$21m^4+82m^3n+82m^2n^2+82mn^3+21n^4 = 0. \qquad (1)$$

Dividing by m^2n^2, $21\dfrac{m^2}{n^2}+82\dfrac{m}{n}+82+82\dfrac{n}{m}+21\dfrac{n^2}{m^2} = 0$.

Let $$y = \frac{m}{n}.$$

Then $$21y^2+82y+82+\frac{82}{y}+\frac{21}{y^2} = 0.$$

This is now a reciprocal equation, and can be solved by the method employed in Ex. 4. A simple solution is obtained by throwing (1) into the form,

$$21(m^2+n^2)^2+82mn(m^2+n^2)+40m^2n^2 = 0.$$

This can now be readily factored.

Ex. 11.—Solve $2x^3 - x^2 - 2x + 1 = 0$.

Add x^4 to both sides.

Then $$x^4 + 2x^3 - x^2 - 2x + 1 = x^4,$$

or $$(x^2 + x - 1)^2 = x^4.$$

Extracting square root, $$x^2 + x - 1 = \pm x^2,$$

$$\therefore\ x - 1 = 0, \text{ or } \quad 2x^2 + x - 1 = 0,$$

$$\therefore\ x = 1, \text{ or } (2x - 1)(x + 1) = 0.$$

Therefore roots are $1, \frac{1}{2}, -1$.

Ex. 12.—Solve $2x^4 - 4x + 1 = 0$.

Multiplying by 2, $$4x^4 - 8x + 2 = 0,$$

or $$4x^4 + 8x^2 + 4 - (8x^2 + 8x + 2) = 0,$$

or $$(2x^2 + 2)^2 - (2\sqrt{2}\,.\,x + \sqrt{2})^2 = 0.$$

Factoring,

$$(2x^2 + 2 - 2\sqrt{2}\,.\,x - \sqrt{2})(2x^2 + 2 + 2\sqrt{2}\,.\,x + \sqrt{2}) = 0.$$

$$\therefore\ 2x^2 + 2 - 2\sqrt{2}\,.\,x - \sqrt{2} = 0, \qquad (1)$$

or $$2x^2 + 2 + 2\sqrt{2}\,.\,x + \sqrt{2} = 0. \qquad (2)$$

From (1) we get $$x = \frac{\sqrt{2} \pm \sqrt{2\sqrt{2} - 2}}{2}.$$

From (2) we get $$x = \frac{-\sqrt{2} \pm \sqrt{-2\sqrt{2} - 2}}{2}.$$

Ex. 13.—Solve $x + \sqrt{x + 1} = 5$.

Rearranging, $$\sqrt{x + 1} = 5 - x.$$

Squaring, $$x + 1 = 25 - 10x + x^2.$$

From which $$x = 3 \text{ or } 8.$$

Upon trial we find that 3 satisfies the given equation, but that 8 belongs to the equation,

$$x - \sqrt{x + 1} = 5.$$

Ex. 14.—Solve $x + \sqrt{7x - 19} = 1$.

Solving as above we obtain the roots 4 and 5, and upon trial we find *both* roots belong to the equation,

$$x - \sqrt{7x - 19} = 1.$$

Had we attempted the solution of this latter equation we should have obtained exactly the same results.

Ex. 15.—Solve $\sqrt{2x + 3} - \sqrt{5x + 1} - 7 = 0$.

Using the result in (2), Art. 197, we have to find its value when a, b and c are the terms of this equation. Arrange this result in the following form:

$$a^2(a^2 - 2b^2 - 2c^2) + (b^2 - c^2)^2 = 0.$$

Here taking $\qquad a^2 = 49,\ b^2 = 2x + 3,\ c^2 = 5x + 1,$

the expression reduces to

$$9x^2 - 698x + 2013 = 0, \tag{1}$$

or

$$(x - 3)(9x - 671) = 0.$$

$$\therefore\ x = 3 \text{ or } 74\tfrac{5}{9}.$$

Upon trial it is found that apparently the original equation is not satisfied by either root, and that the root 3 belongs to the equation,

$$\sqrt{2x + 3} + \sqrt{5x + 1} - 7 = 0, \tag{2}$$

and the root $74\frac{5}{9}$ to the equation,

$$\sqrt{2x + 3} - \sqrt{5x + 1} + 7 = 0. \tag{3}$$

The difficulty may be explained in two ways:

(a) It has already been explained in Part I. that the sign of the square root of a quantity may be either positive or negative. If we take *positive* signs of the roots of the surds in the original equation, the value 3 will not satisfy the equation; but if we take the *negative* sign of the roots in $\sqrt{5x + 1}$, and the positive

sign in $\sqrt{2x+3}$, the equation will be satisfied by $x=3$. A similar explanation applies to the root $74\frac{5}{9}$.

(*b*) The difficulty may also be explained by pointing out that equation (1) is the product of four factors, of which (2) and (3) are two, and consequently any value which satisfies either (2) or (3) must satisfy (1).

A similar explanation applies to Exs. 13 and 14.

If we are restricted to the *positive* root in each case, the given equation has no solution.

Ex. 16.—An integral equation of the third degree can always be expressed in the form

$$x^3+ax^2+bx+c=0.$$

Let
$$x=y-\frac{a}{3}.$$

Substituting, and arranging in powers of y, we get

$$y^3+\left(b-\frac{a^2}{3}\right)y+\left(\frac{2a^2}{27}-\frac{ab}{3}\right)=0.$$

Now, if this equation can be solved we shall have the roots of the original equation; for $x=y-\frac{a}{3}$, and therefore when x is known y is known. If, therefore, we can solve a cubic equation in which the term containing x^2 is wanting, we can solve any cubic equation.

Ex. 17.—Solve the equation, $x^3-qx-r=0$.

Let
$$x=y+\frac{q}{3y}.$$

Then
$$x^3=y^3+\frac{q^3}{27y^3}+\frac{3yq}{3y}\left(y+\frac{q}{3y}\right)$$

$$=y^3+\frac{q^3}{27y^3}+qx,$$

or
$$x^3-qx=y^3+\frac{q^3}{27y^3}=r.$$

Then $$y^6 - ry^3 + \frac{q^3}{27} = 0,$$

from which $$y^3 = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}$$

and $$x = y + \frac{q}{3y}$$

$$= \left\{\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right\}^{\frac{1}{3}} + \left\{\frac{r}{2} \mp \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right\}^{\frac{1}{3}}$$

after rationalizing the denominator and simplifying the second term.

It should be observed that the same result is obtained by taking either the upper or the lower signs with the radical. But every quantity has three cube roots, and we must determine which are admissible. For brevity, denote the first term by p and the second by q, and the cube root of unity by ω; the roots will then be

$$p + q, \ p\omega + q\omega^2, \ p\omega^2 + q\omega,$$

as may easily be verified by trial.

Ex. 18.—Solve $x^3 - \sqrt[3]{6}\,.\,x = 1$.

Let $$x = y + \frac{\sqrt[3]{6}}{3y}.$$

Then $$x^3 = y^3 + \frac{2}{9y^3} + \sqrt[3]{6}\,.\,x$$

or $$x^3 - \sqrt[3]{6}\,.\,x = y^3 + \frac{2}{9y^3} = 1,$$

from which $$y^3 = \frac{1}{3} \text{ or } \frac{2}{3}.$$

Then $$x = y + \frac{\sqrt[3]{6}}{3y}$$

$$= \sqrt[3]{\frac{1}{3}} + \sqrt[3]{\frac{2}{3}}, \ \omega\sqrt[3]{\frac{1}{3}} + \omega^2\sqrt[3]{\frac{2}{3}}, \text{ or } \omega^2\sqrt[3]{\frac{1}{3}} + \omega\sqrt[3]{\frac{2}{3}},$$

EXERCISE XVII.

Solve the following:

1. $\dfrac{a(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{b(x-c)(x-a)}{(b-c)(b-a)}=x.$

2. $\dfrac{(a-x)^2+(x-b)^2}{(a-x)(x-b)}=\dfrac{5}{2}.$

3. $\dfrac{x}{x^2-2x-15}-\dfrac{7.5}{x^2+2x-35}=\dfrac{1}{x^2+10x+21}.$

4. $\dfrac{1}{21x^2-13x+2}+\dfrac{1}{28x^2-15x+2}=12x^2-7x+1.$

5. $\dfrac{8}{x^2-6x+5}+\dfrac{8}{x^2-14x+45}=\dfrac{x^4}{x^2-10x+9}.$

6. $\dfrac{5}{x^2-7x+10}+\dfrac{5}{x^2-13x+40}=x^2-10x+19.$

7. $\dfrac{(a-x)^2+(a-x)(x-b)+(x-b)^2}{(a-x)^2-(a-x)(x-b)+(x-b)^2}=\dfrac{49}{19}.$

8. $\dfrac{1}{(x+a)^2-b^2}+\dfrac{1}{(x+b)^2-a^2}=\dfrac{1}{x^2-(a+b)^2}+\dfrac{1}{x^2-(a-b)^2}.$

9. $(x-7)(x-3)(x+5)(x+1)=1680.$

10. $16x(x+1)(x+2)(x+3)=9.$

11. $\sqrt{x^2-a^2-b^2}+\sqrt{x^2-b^2-c^2}-\sqrt{x^2-c^2-a^2}=x.$

12. $\dfrac{\sqrt{x+2}+\sqrt{x+1}}{\sqrt{x+2}-\sqrt{x+1}}+\dfrac{\sqrt{x+2}-\sqrt{x+1}}{\sqrt{x+2}+\sqrt{x+1}}=3x(x-1).$

13. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}=4\sqrt{x^2-1}.$

14. $\dfrac{x}{a+x}+\dfrac{a}{\sqrt{a+x}}=\dfrac{b}{x}.$

15. $\sqrt{3+\sqrt{x}}+\sqrt{4-\sqrt{x}}=\sqrt{7+2\sqrt{x}}.$

16. $\sqrt[m]{(a+x)^2}-\sqrt[m]{(a-x)^2}=\sqrt[m]{a^2-x^2}.$

17. $\sqrt[3]{(1+x)^2}-\sqrt[3]{(1-x^2)}-2\sqrt[3]{(1-x)^2}=0.$

18. $\sqrt{3x^2-2x+9}+\sqrt{3x^2-2x-4}=13.$

19. $\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}=x-5.$

20. $\sqrt{x^2+2x-1}+\sqrt{x^2+x+1}=\sqrt{2}+\sqrt{3}.$

21. $\sqrt{x^2+ax+b^2}+\sqrt{x^2+bx+a^2}=a+b.$

22. $(x+2\sqrt{x})^{\frac{1}{2}}-(x-2\sqrt{x})^{\frac{1}{2}}=2(x^2-4x)^{\frac{1}{8}}.$

23. $\sqrt{a^2-x^2}+x\sqrt{a^2-1}=a^2\sqrt{1-x^2}.$

24. $\frac{1}{5}\left(\frac{x^2-2x-3}{x^2-2x-8}\right)+\frac{1}{9}\left(\frac{x^2-2x-15}{x^2-2x-24}\right)-\frac{2}{13}\left(\frac{x^2-2x-35}{x^2-2x-48}\right)=\frac{92}{585}.$

25. $x^3+px^2+px+1=0.$

26. $10^{(x-1)(2-x)}=1000.$

27. $x^4-4x^3+6x^2-4x-15=0.$

28. $x^4+5x^3+8x^2+5x+1=0.$

29. $3^{2x}+9=10\times 3^x.$

30. $2^{2x+8}+1=32\times 2^x.$

31. $2^{2x+3}-57=65(2^x-1).$

32. $x^4+\frac{8}{9}x^2+1=3x^3+3x.$

33. $(a+x)^{\frac{2}{3}}+4(a-x)^{\frac{2}{3}}=5(a^2-x^2)^{\frac{1}{3}}.$

34. $x^4-2x^3+x=380.$

35. $27x^3+21x+8=0.$

36. $a^{2x}(a^2+1)=(a^{3x}+a^x)a.$

37. $\frac{(a-x)^5+(x-b)^5}{(a-x)^4+(x-b)^4}=\frac{211}{97}(a-b).$

38. $\frac{(a-x)^4-(x-b)^4}{(a-x)^2-(x-b)^2}=\frac{(a-b)c}{(a-x)(x-b)}.$

39. $x^3-6x^2+5x+12=0.$

40. $\frac{(x+1)^5}{x^5+1}=\frac{a}{b}.$

41. $x^4+ax^3+bx^2+cx+\frac{c^2}{a^2}=0.$

42. $\frac{x^2-5}{x^2-6}+\frac{x^2-11}{x^2-12}=\frac{x^2-7}{x^2-8}+\frac{x^2-9}{x^2-10}.$

43. $x+a+3\sqrt[3]{abx}=b.$

44. $\dfrac{x^{(m-n)^2}+x^{-4mn}}{x^{(m-n)^2}-x^{-4mn}}=a^{\frac{r}{s}}.$

45. $x^{\frac{p+q}{2pq}}-\dfrac{1}{2}\dfrac{a^2-b^2}{a^2+b^2}\left(x^{\frac{1}{p}}+x^{\frac{1}{q}}\right)=0.$

46. $a^2b^2x^{\frac{1}{n}}-4(ab)^{\frac{3}{2}}x^{\frac{m+n}{2mn}}=(a-b)^2.x^{\frac{1}{m}}.$

47. $(x^2-1)^{\frac{1}{2}}+(x^4-1)^{\frac{1}{2}}=x^3.$

48. $x^4-6x^3+5x^2+8x-4=0.$

49. $x^4=4(x-1)(1-x-x^2).$

50. $1+4x-8x^2+2x^4=0.$

51. $(5x^2+x+10)^2+(x^2+7x+1)^2=(3x^2-x+5)^2+(4x^2+5x+8)^2.$

52. $(12x-1)(6x-1)(4x-1)(3x-1)=\dfrac{1}{96}.$

53. $8x^{\frac{3}{4}}+81=18x^{\frac{1}{2}}+45x^{\frac{1}{4}}.$

54. $x^3+3x=a^3-\dfrac{1}{a^3}.$

55. $(1+x^8)+(1+x)^8=2(1+x+x^2)^4.$

56. $x^4-2x^3-3x^2-12x+36=0.$

57. $x^4-8x^3+10x^2+24x+5=0.$

58. $(x+b+c)(x+c+a)(x+a+b)=(x+a)(x+2b)(x+3c).$

59. $\dfrac{40}{x^2+2x-48}-\dfrac{20}{x^2+9x+8}-\dfrac{8}{x^2+10x}+\dfrac{12}{x^2+5x-50}+1=0.$

60. $\dfrac{(x-b)(x-c)a^2}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)b^2}{(b-c)(b-a)}+\dfrac{(x-a)(x-b)c^2}{(c-a)(c-b)}=x^2.$

61. $(x^3-2x^2-2x+3)(x^3-4x^2+4x-3)$
$$=(x^3+2x^2-2x-3)(x^3+4x^2+4x+3).$$

62. $x^4+2ax^3+\dfrac{(b^2-a^2)^3}{4b^2}=0$ [by putting it in the form
$$\left(x^2+ax+\frac{b^2-a^2}{2}\right)^2=\left\{bx+\frac{a(b^2-a^2)}{2b}\right\}^2\Bigg].$$

63. $x^4-8x^3-108=0.$

64. $x^4-10x^3-3456=0.$

65. $x^3-18x-35=0.$

66. $x^3+72x-1720=0.$

67. $x^3-15x^2-33x+847=0.$

68. $2x^3+25x^2+56x-147=0.$

69. $8x^3-36x+27=0.$

CHAPTER XIII.

SIMULTANEOUS EQUATIONS OF THE SECOND AND HIGHER DEGREES.

TWO OR MORE UNKNOWNS.

199. In Part I., Chapter XIX., illustrations were given of simultaneous quadratic equations of two unknowns. The following are examples of simultaneous equations of a more difficult character. The student will observe that no general rule can be given for the solution of such equations. A careful study of the following examples will enable him to solve all the more important and useful problems that may arise.

200. Before proceeding to give examples of problems of the *second* and higher degrees we will give some explanation of a method, not introduced in Part I., of solving simultaneous equations of the *first* degree. This method is called that of **Indeterminate Multipliers,** and is best explained by an illustration:

Ex.—Solve

$$ax + by + cz = d, \qquad (1)$$

$$a_1x + b_1y + c_1z = d_1, \qquad (2)$$

$$a_2x + b_2y + c_2z = d_2, \qquad (3)$$

Multiplying (1) and (2) by l and m, and adding (3), and arranging the result, viz.:

$$x(la + ma_1 + a_2) + y(lb + mb_1 + b_2) + z(lc + mc_1 + c_2) = ld + md_1 + d_2.$$

Now let l and m have such values that the coefficients of y and z may both be zero.

Then
$$x = \frac{ld + md_1 + d_2}{la + ma_1 + a_2},$$
where
$$lb + mb_1 + b_2 = 0$$
and
$$lc + mc_1 + c_2 = 0.$$

$$\therefore \quad \frac{l}{b_1c_2 - b_2c_1} = \frac{m}{b_2c - bc_2} = \frac{1}{bc_1 - b_1c}.$$

Substituting these values of l and m in

$$\frac{ld + md_1 + d_2}{la + ma_1 + a_2}$$

we obtain
$$x = \frac{d(b_1c_2 - b_2c_1) + d_1(b_2c - bc_2) + d_2(bc_1 - b_1c)}{a(b_1c_2 - b_2c_1) + a_1(b_2c - bc_2) + a_2(bc_1 - b_1c)}.$$

Having thus found the value of x, the values of y and z can be written down by symmetry.

It is evident that this method may be employed when more than *three* unknown quantities are given, all that is necessary being a corresponding increase in the number of indeterminate multipliers. The student may, as an exercise on this method, take any ordinary selection of problems in simultaneous equations, and find the solutions required.

201. We proceed now with the consideration of the subject-matter proper of this chapter.

202. When both equations are homogeneous and of the same number of dimensions, the method of elimination may be employed.

Ex.—Solve
$$y^3 - 3xy^2 + 5x^2y = 15, \tag{1}$$
$$xy^2 - 2x^2y + 2x^3 = 5, \tag{2}$$

Multiplying (2) by 3 and subtracting from (1) we get

$$y^3 - 6xy^2 + 11x^2y - 6x^3 = 0.$$

Factoring, $(y-x)(y-2x)(y-3x)=0.$

Therefore $y=x,\ 2x \text{ or } 3x.$

Substituting these values successively in (1) we get

$$x^3=5,\ \frac{5}{2} \text{ or } 1.$$

Then $x=\sqrt[3]{5},\ \sqrt[3]{\frac{5}{2}} \text{ or } 1,$

and $y=\sqrt[3]{5},\ \sqrt[3]{20} \text{ or } 3.$

Each pair of roots may be multiplied by an imaginary cube root of unity, giving six other solutions.

203. It is sometimes convenient to find the values of $x+y$ and xy before finding the value of each letter separately.

Ex.—Solve
$$x^2+y^2+x+y=18, \qquad (1)$$
$$6(x+y)=5xy. \qquad (2)$$

Equation (1) may be written

$$(x+y)^2+(x+y)-2xy=18.$$

Substituting for xy from (2),

$$(x+y)^2-\frac{7}{5}(x+y)-18=0, \qquad (3)$$

from which $x+y=5 \text{ or } -\frac{18}{5}; \qquad (4)$

then from (2), $xy=6 \text{ or } -\frac{108}{25}. \qquad (5)$

Squaring (4) and subtracting four times (5),

$$x-y=\pm 1 \text{ or } \pm\frac{6}{5}\sqrt{21}. \qquad (6)$$

Combining (4) and (6), $x=3,\ 2 \text{ or } \frac{3}{5}(-3\pm\sqrt{21}),$

$$y=2,\ 3 \text{ or } \frac{3}{5}(-3\mp\sqrt{21}).$$

When the values of $x+y$ and xy have been obtained, the values of the separate letters may be neatly written from the quadratic whose *second term is the sum of x and y with the sign changed, and whose product is the last term*, a different letter being used as variable. Thus from the preceding example we have

$$r^2-5r+6=0 \text{ and } r^2+\frac{18}{5}r-\frac{108}{25}=0,$$

from which $\qquad r=2 \text{ or } 3, \text{ or } \qquad r=\frac{3}{5}(-3\pm\sqrt{21}).$

The two values of r derived from either equation will give two solutions, one value being given to x, and the other to y.

204. When one solution of a pair of simultaneous equations has been found, other solutions may frequently be written at once from the following considerations:

1. If the variables are symmetrically involved, their values may be interchanged. (See Art. 203.)

2. If each term is of an even number of dimensions, the signs of both values may be changed.

3. If each exponent is even, the sign of each value may be separately changed.

4. If the literal part of each equation changes signs when the variables are interchanged, and if each term is of an odd number of dimensions, the values may be interchanged, providing both the signs are also changed.

205.—Another artifice sometimes used is the finding of the values of $x+y$ and $x-y$ before finding the values of x and y.

Ex.—Solve $\qquad (x+y)(x^3+y^3)=76, \qquad (1)$

$$(x+y)^3=64(x-y). \qquad (2)$$

Assume $\qquad x+y=m \text{ and } x-y=n,$

$$\therefore\ x=\frac{m+n}{2} \text{ and } y=\frac{m-n}{2}.$$

Then
$$\begin{aligned} x^3+y^3 &= (x+y)(x^2-xy+y^2) \\ &= m\{(x-y)^2+xy\} \\ &= m\left(n^2+\frac{m^2-n^2}{4}\right) \\ &= \frac{m(3n^2+m^2)}{4}. \end{aligned}$$

Equations (1) and (2) now become
$$m^2(3n^2+m^2)=304, \tag{3}$$
and
$$m^3=64n. \tag{4}$$

Dividing (3) by (4), $$\frac{3n^2+m^2}{m}=\frac{19}{4n},$$

or $$4n(3n^2+m^2)=19m. \tag{5}$$

But from (4), $$n=\frac{m^3}{64};$$

$$\therefore \frac{m^3}{16}\left(\frac{3m^6}{64^2}+m^2\right)=19m,$$

or $$\frac{3m^8}{16\times 64^2}+\frac{m^4}{16}=19,$$

or $$\frac{3m^8}{16^2\times 16^2}+\frac{m^4}{16}=19. \tag{6}$$

Let $$\frac{m^4}{4^4}=z.$$

Then (6) becomes $$3z^2+16z=19,$$

or
$$\begin{aligned} 3z^2+16z-19 &= 0 \\ =(3z+19)(z-1) &= 0. \end{aligned}$$
$$\therefore z=1 \text{ or } -\frac{19}{3}.$$

Solutions can now be readily obtained.

206. The following are further illustrations of methods:

Ex. 1.—Solve $$x^3 = 31x^2 - 4y^2, \qquad (1)$$
$$y^3 = 31y^2 - 4x^2. \qquad (2)$$

Dividing (1) by (2), $$\frac{x^3}{y^3} = \frac{31x^2 - 4y^2}{31y^2 - 4x^2}. \qquad (3)$$

Let $$\frac{x}{y} = m.$$

Then $$m^3 = \frac{31m^2 - 4}{31 - 4m^2}, \qquad (4)$$

or $$31m^3 - 4m^5 = 31m^2 - 4, \qquad (5)$$

or $$31(m^3 - m^2) = 4(m^5 - 1), \qquad (6)$$

or $$31m^2(m-1) = 4(m-1)(m^4 + m^3 + m^2 + m + 1). \qquad (7)$$

It is evident the equation (7) is satisfied by $m = 1$, or $\frac{x}{y} = 1$, or $x = y$. If $x = y$, then $x = y = 27$.

The equation left after $m - 1$ is struck out is

$$31m^2 = 4(m^4 + m^3 + m^2 + m + 1),$$

or $$31 = 4\left(m^2 + m + 1 + \frac{1}{m} + \frac{1}{m^2}\right),$$

or $$31 = 4\left(m + \frac{1}{m^2}\right) + 4\left(m + \frac{1}{m}\right) + 4,$$

or $$35 = 4\left(m + \frac{1}{m}\right)^2 + 4\left(m + \frac{1}{m}\right).$$

Let $$m + \frac{1}{m} = z.$$

Then $$35 = 4z^2 + 4z,$$

from which we obtain

$$2z + 1 = \pm 6, \text{ or } 2z = 5 \text{ or } -7.$$

If $m+\frac{1}{m}$ is now substituted for z, and then $\frac{x}{y}$ for m, we obtain x in terms of y, and find the values of x and y to be

$$\frac{7}{2}(3\pm\sqrt{33}) \text{ and } \frac{7}{2}(3\mp\sqrt{33}).$$

Other values are, $\left.\begin{array}{l} x=15,\ 30, \\ y=30,\ 15. \end{array}\right\}$

Ex. 2.—Solve $$(x+y)^{\frac{1}{3}}+(x-y)^{\frac{1}{3}}=a^{\frac{1}{3}}, \tag{1}$$

$$(x^2+y^2)^{\frac{1}{3}}+(x^2-y^2)^{\frac{1}{3}}=a^{\frac{2}{3}}. \tag{2}$$

Cubing (1), $x+y+x-y+3(x^2-y^2)^{\frac{1}{3}}a^{\frac{1}{3}}=a,$

or $$2x+3(x^2-y^2)^{\frac{1}{3}}a^{\frac{1}{3}}=a,$$

or $$3(x^2-y^2)^{\frac{1}{3}}a^{\frac{1}{3}}=a-2x,$$

or $$(x^2-y^2)^{\frac{1}{3}}=\frac{a-2x}{3a^{\frac{1}{3}}}. \tag{3}$$

Cubing (2) and treating the equation in a similar fashion to (1),

$$(x^4-y^4)^{\frac{1}{3}}=\frac{a^2-2x^2}{3a^{\frac{2}{3}}}. \tag{4}$$

Dividing (4) by (3), $$(x^2+y^2)^{\frac{1}{3}}=\frac{a^2-2x^2}{a-2x}\cdot\frac{1}{a^{\frac{1}{3}}}. \tag{5}$$

Substituting these values in (2) we get

$$\frac{a-2x}{3a^{\frac{1}{3}}}+\frac{a^2-2x^2}{a^{\frac{1}{3}}(a-2x)}=a^{\frac{2}{3}}. \tag{6}$$

(6) can now be readily solved as an ordinary quadratic. The values of x and y are:

$$x=\frac{a}{2}(1\pm\sqrt{3}),\quad y=\frac{a}{2}\left(1-\frac{1}{\sqrt{3}}\right)\left(\sqrt{1-\frac{4}{\sqrt{3}}}\right).$$

Ex. 3.—Solve
$$x^2 = ax + by, \tag{1}$$
$$y^2 = bx + ay. \tag{2}$$

Subtracting (2) from (1),
$$x^2 - y^2 = (a - b)(x - y). \tag{3}$$

The equation is satisfied by putting $x - y = 0$ or $x = y$. If $x = y$, then
$$x^2 = ax + bx, \text{ and } x = 0 \text{ or } a + b;$$
$$\therefore\ y = 0 \text{ or } a + b.$$

Dividing (3) by $(x - y)$ we get
$$x + y = a - b \text{ or } y = a - b - x.$$

Substituting this value of y in (1) or (2) we find
$$x = \frac{1}{2}\{(a - b) \pm \sqrt{(a - b)(a + 3b)}\}$$

and therefore
$$y = \frac{1}{2}\{(a - b) \mp \sqrt{(a - b)(a + 3b)}\}.$$

Ex. 4.—Solve
$$x^4 + y^4 = 706, \tag{1}$$
$$x + y = 8. \tag{2}$$

From (2),
$$(x + y)^4 = 4096,$$
or
$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 4096.$$
But
$$x^4 + y^4 = 706, \tag{1}$$
$$\therefore\ 4x^3y + 6x^2y^2 + 4xy^3 = 3390,$$
or
$$2x^3y + 3x^2y^2 + 2xy^3 = 1695,$$
or
$$xy(2x^2 + 3xy + 2y^2) = 1695. \tag{3}$$

Now,
$$\begin{aligned} 2x^2 + 3xy + 2y^2 &= 2(x + y)^2 - xy \\ &= 2(8)^2 - xy \qquad \text{from (2)} \\ &= 128 - xy. \end{aligned}$$

Substituting in (3), $\qquad xy(128 - xy) = 1695,$

or $$x^2y^2 - 128xy + 1695 = 0, \tag{4}$$

or $$(xy - 15)(xy - 113) = 0;$$

$$\therefore\ xy = 15 \text{ or } 113.$$

Combining these results with $x + y = 8$ we readily find the values of x and y. One set of values is

$$\left.\begin{aligned} x &= 3,\ 5, \\ y &= 5,\ 3. \end{aligned}\right\}$$

EXERCISE XVIII.

Solve the following equations:

1. $x^3 + y^3 = 65,$
   $x + y = 5.$

2. $x^4 + y^4 = 2417,$
   $x + y = 9.$

3. $x^5 - y^5 = 16564,$
   $x - y = 4.$

4. $x^4 + y^4 = 641,$
   $x - y = 3.$

5. $x^3 + y^3 = 35,$
   $x^2 + y^2 = 13.$

6. $x^4 - y^4 = 65,$
   $x - y = 1.$

7. $x^4 + x^2y^2 + y^4 = 133,$
   $x^2 + xy + y^2 = 19.$

8. $x^2 + xy + y^2 = 91,$
   $x + \sqrt{xy} + y = 13.$

9. $x^2 + xy + y = 37,$
   $y^2 + xy + x = 19.$

10. $x^4 + 4y^4 = a^4,$
    $x^2 + 2xy + 2y^2 = b^2.$

11. $(x + y)(x^3 + y^3) = 1216,$
    $x^2 + xy + y^2 = 49.$

12. $x^3 + y^3 = a^3,$
    $x^2y + xy^2 = b^3,$

13. $\dfrac{(x + y)^2}{a^2} + \dfrac{(x - y)^2}{b^2} = m,$
    $x^2 + y^2 = \dfrac{1}{2}n.$

14. $\dfrac{a^2}{(x + y)^2} - \dfrac{b^2}{(x - y)^2} = \dfrac{ma^2b^2}{(x^2 - y^2)^2},$
    $x^2 + y^2 = \dfrac{1}{2}n.$

15. $x^y = y^x,$
$x^2 = y^3.$

16. $x^2y = a^3,$
$xy^2 = b^3.$

17. $x^y = y^x,$
$x = 2y.$

18. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5,$
$x^{-\frac{1}{2}} + y^{-\frac{1}{2}} = \frac{5}{6}.$

19. $x(x^2 + y^2) = 6y,$
$y(x^2 - y^2) = x.$

20. $\frac{x}{y} - \frac{y}{x} = \frac{x+y}{x^2+y^2},$
$\frac{x^2}{y^2} - \frac{y^2}{x^2} = \frac{x-y}{y^2}.$

21. $x + xy^3 = 18,$
$xy + xy^2 = 12.$

22. $x^2 + x\sqrt[3]{xy^2} = 208,$
$y^2 + y\sqrt[3]{x^2y} = 1053.$

23. $\left(\frac{x}{y} + 1\right)\left(y + \frac{1}{x}\right) = 5\frac{1}{3},$
$\left(\frac{x^2}{y^2} + 1\right)\left(y^2 + \frac{1}{x^2}\right) = 11\frac{1}{9}.$

24. $x + y = xy = x^2 - y^2.$

25. $(x^2 + y^2)(x^3 + y^3) = 455,$
$x + y = 5.$

26. $x^4 + y^4 = 641,$
$x^3y + xy^3 = 290.$
(Find 16 solutions.)

27. $x - y = a,$
$x^4 + y^4 = 18x^2y^2.$

28. $\frac{x+y}{1-xy} = 2,$
$\frac{1+xy}{x-y} = 2.$

29. $x^2 + y = 7,$
$y^2 + x = 11.$

30. $\frac{x}{x+y} + \frac{y}{x-y} = \frac{n^2+m^2}{n^2-m^2},$
$\frac{x}{x-y} + \frac{y}{x+y} = \frac{n^2+2mn-m^2}{n^2-m^2}.$

31. $x^5 + y^5 = 178\sqrt{3}.$
$x^2 + y^2 = 10xy.$

32. $x^{x^2+y^2} = y^{\frac{8}{9}},$
$y^{x^2+y^2} = x^{\frac{2}{9}}.$

33. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = \frac{1}{15}(xy^{\frac{1}{2}} + x^{\frac{1}{2}}y) = 8.$

34. $\frac{ax}{a+x} + \frac{by}{b+y} = \frac{(a+b)c}{a+b+c}.$
$x + y = c.$

35. $\frac{a(y+c)}{x+c} + \frac{b(x+c)}{y+c} = a + b,$
$bx - ay = c(a - b).$

36. $x^2y - 4 = 4x^{\frac{1}{2}}y - \frac{1}{4}y^3.$
$x^{\frac{3}{2}} - 3 = x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}}).$

37. $x^3 + y^3 + xy(x + y) = 13,$
$(x^2 + y^2)(x^2y^2) = 468.$

38. $\sqrt{ax} + \sqrt{by} = a + b,$
$x + y = 2(a + b).$

39. $\sqrt{x+y} + \sqrt{x-y} = 4,$
$x^2 - y^2 = 9.$

40. $x^4 = mx + ny,$
$y^4 = my + nx.$

41. $(x^2 + y^2)\frac{y}{x} = \frac{26}{3},$
$(x^2 - y^2)\frac{x}{y} = \frac{15}{2}.$

42. $(x^2 + y^2)(x + y) = 15xy,$
$(x^4 + y^4)(x^2 + y^2) = 85x^2y^2.$

43. $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{5}{2},$
$\sqrt{\frac{x^2}{y}} + \sqrt{\frac{y^2}{x}} = \frac{9\sqrt{2}}{2}.$

44. $y^3 + 11x^2y = 480,$
$x^3 + xy^2 = 80.$

45. $\sqrt{(x^2 + a^2)(y^2 + b^2)} + \sqrt{(x^2 + b^2)(y^2 + a^2)} = (a + b)^2,$
$x + y = a + b.$

207. The following are examples of simultaneous equations of *three* unknowns:

Ex. 1.—Solve

$$x + y + z = 4, \quad (1)$$

$$x^2 + y^2 + z^2 = 14, \quad (2)$$

$$xy + yz - zx = 5. \quad (3)$$

Square (1) and subtract (2) from it, and divide by 2,

$$xy + yz + zx = 1. \tag{4}$$

Combining (3) and (4),

$$y(x+z) = 3, \quad zx = -2. \tag{5}$$

From (1), $x + z = 4 - y$.

Substituting in (5), $y(4-y) = 3$,

or $y^2 - 4y + 3 = 0$,

from which $y = 3$ or 1.

If $y = 3$, $x + z = 1$, and from (5), $zx = -2$.

Therefore $x = 2$ or -1 and $z = -1$ or 2.

Similarly, when $y = 1$, $x = \frac{1}{2}(3 \pm \sqrt{17})$, $z = \frac{1}{2}(3 \mp \sqrt{17})$.

From the above we see that y has two values, whilst x and z have each four values, and it is necessary that these values should be correctly grouped to give a true solution. The correct arrangement is:

$$\begin{array}{rlll} x = & 2, & -1, & \frac{1}{2}(3 \pm \sqrt{17}); \\ y = & 3, & 3, & 1; \\ z = & -1, & 2, & \frac{1}{2}(3 \mp \sqrt{17}). \end{array}$$

Ex. 2.—Solve $xy + yz + zx = a^2 - x^2 = b^2 - y^2 = c^2 - z^2$.

Rearranging and factoring we get

$$(x+y)(x+z) = a^2,$$

$$(y+z)(y+x) = b^2,$$

$$(z+x)(z+y) = c^2.$$

Multiplying the three equations together and taking the square root we get

$$(x+y)(y+z)(z+x)=\pm abc.$$

Dividing this result by each of the original equations in succession we get

$$x+y=\pm\frac{bc}{a},\quad y+z=\pm\frac{ca}{b},\quad z+x=\pm\frac{ab}{c}.$$

Adding these equations and dividing by 2 we get

$$x+y+z=\pm\frac{a^2b^2+b^2c^2+c^2a^2}{2abc}.$$

Thence by subtraction, $\quad x=\pm\dfrac{b^2c^2+c^2a^2-a^2b^2}{2abc},$

from which the values of y and z may be written from symmetry.

Ex. 3.—Solve

$$x+y+z=10, \tag{1}$$

$$xy+yz+zx=31, \tag{2}$$

$$xyz=30. \tag{3}$$

Equation (2) may be written

$$xy+z(x+y)=31. \tag{4}$$

From (1) and (3), $\quad x+y=10-z,\quad xy=\dfrac{30}{z}. \qquad (5)$

Substituting these values in (4),

$$\frac{30}{z}+z(10-z)=31.$$

Rearranging, $\quad z^3-10z^2+31z-30=0. \qquad (6)$

Factoring, $\quad (z-2)(z-3)(z-5)=0,$

from which $\quad z=2,\ 3$ or $5.$

The values of x and y are then easily found.

The following solution is worthy of attention:

Consider the following equations in r:

$$(r-x)(r-y)(r-z)=0,$$

or $$r^3-(x+y+z)r^2+(xy+yz+zx)r-xyz=0,$$

Its three roots are evidently x, y and z. If, then, for

$$x+y+z, \quad xy+yz+zx \text{ and } xyz$$

we substitute their values from the original equations, we shall obtain a cubic equation in r whose roots are the values of x, y and z. The equation will be

$$r^3-10r^2+31r-30=0,$$

which is identical with (6) previously obtained. The three values of r, viz., 2, 3 and 5, may be assigned to x, y and z in *six* different ways as follows:

$$\begin{aligned} x &= 2, \ 2, \ 3, \ 3, \ 5, \ 5; \\ y &= 3, \ 5, \ 2, \ 5, \ 2, \ 3; \\ z &= 5, \ 3, \ 5, \ 2, \ 3, \ 2. \end{aligned}$$

The methods of this solution are of great importance in the application of Algebra to Geometry.

Ex. 4.—Solve

$$(x-y)(y-z)=a^2, \tag{1}$$
$$(y-z)(z+x)=b^2, \tag{2}$$
$$(z+x)(x+y)=c^2. \tag{3}$$

Divide (1) by (2) and (2) by (3), and simplify.

$$(b^2-a^2)x-b^2y-a^2z=0, \tag{4}$$
$$b^2x+(b^2-c^2)y+c^2z=0. \tag{5}$$

From (4) and (5),

$$\frac{x}{a^2b^2-b^2c^2-c^2a^2}=\frac{y}{c^2a^2-a^2b^2-b^2c^2}=\frac{z}{2b^4-b^2c^2+c^2a^2-a^2b^2}$$

$$=\frac{x-y}{2a^2(b^2-c^2)}=\frac{y-z}{-2b^4}. \tag{6}$$

Substituting from (6) in (1),

$$-4a^2b^4(b^2-c^2)=\frac{a^2(a^2b^2-b^2c^2-c^2a^2)^2}{x^2},$$

from which
$$x=\frac{a^2b^2-b^2c^2-c^2a^2}{2b^2\sqrt{c^2-b^2}};$$

then from (6),
$$y=\frac{c^2a^2-a^2b^2-b^2c^2}{2b^2\sqrt{c^2-b^2}},$$

and
$$z=\frac{2b^4+c^2a^2-a^2b^2-b^2c^2}{2b^2\sqrt{c^2-b^2}}.$$

Ex. 5.—Solve
$$x^2(y-z)=a^3, \quad (1)$$
$$y^2(z-x)=b^3, \quad (2)$$
$$z^2(x-y)=c^3. \quad (3)$$

Adding the equations and factoring,

$$-(x-y)(y-z)(z-x)=a^3+b^3+c^3. \quad (4)$$

Multiplying the three equations and dividing by (4),

$$x^2y^2z^2=-\frac{a^3b^3c^3}{a^3+b^3+c^3}=R^2.$$

Then
$$xyz=R. \quad (5)$$

Dividing (1), (2) and (3) by x, y and z, and adding,

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}=0,$$

or
$$a^3yz+b^3zx+c^3xy=0. \quad (6)$$

Eliminating x from (5) and (6),

$$a^3y^2z^2+R(b^3z+c^3y)=0. \quad (7)$$

Eliminating x from (1) and (5),

$$R^2(y-z)=a^3y^2z^2. \quad (8)$$

From (7) and (8),

$$R(y-z)+b^3z+c^3y=0,$$

from which
$$y=\left(\frac{R-b^3}{R+c^3}\right)z. \tag{9}$$

Substituting from (9) in (8),

$$R^2\left(\frac{R-b^3}{R+c^3}-1\right)z=a^3\left(\frac{R-b^3}{R+c^3}\right)^2z^4,$$

from which
$$z^3=-\frac{R^2(b^3+c^3)(R+c^3)}{a^3(R-b^3)^2}.$$

Then
$$x^3=-\frac{R^2(c^3+a^3)(R+a^3)}{b^3(R-c^3)^2},$$

and
$$y^3=-\frac{R^2(a^3+b^3)(R+b^3)}{c^3(R-a^3)^2}.$$

The values of x^3 and y^3 are written from that of z^3 by symmetry.

208. In the application of Algebra to Geometry symmetrical expressions with three letters frequently occur. They usually arise from the sides or the angles of a triangle, or from the three dimensions of space. A knowledge of the more usual forms and facility in making transformations is desirable. The solution of the following equations will furnish exercise in such work. The following identities will sometimes be found useful:

$$x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+zx).$$

$$x^3+y^3+z^3=(x+y+z)^3-3(x+y+z)(xy+yz+zx)+3xyz.$$

$$x^2(y+z)+y^2(z+x)+z^2(x+y)=(x+y+z)(xy+yz+zx)-3xyz$$

EXERCISE XIX.

Solve the equations:

1. $x+y+z=6,\quad x^2+y^2+z^2=26,\quad xy-yz+zx=13.$

2. $x+y+z=8,\quad x^2+y^2+z^2=90,\quad yz+zx-xy=43.$

3. $x+y-z=1,\quad x^2+y^2-z^2=-15,\quad xy=15.$

4. $2x+3y-z=6,\quad 4x^2-9y^2+z^2=16,\quad xz=-2.$

5. $x+y+z=0,\quad xy+yz+zx=-7,\quad xyz=6.$

6. $x+y+z=2,\quad xy+yz+zx=-23,\quad xyz=-60.$

7. $x+y+z=5,\quad x^3+y^3+z^3=965,\quad xyz=60.$

8. $x+y+z=-1,\quad x^2+y^2+z^2=35,\quad x^3+y^3+z^3=-97.$

9. $x^2(y+z)+y^2(z+x)+z^2(x+y)=22,\quad x+y+z=4,\quad xyz=-6.$

10. $x+y+z=9,\quad xy+yz+zx=26,\quad (y+z)(z+x)(x+y)=210.$

11. $x^2+xy+xz=18,\quad y^2+yz+yx=-30,\quad z^2+zx+zy=48.$

12. $xy+x+y=7,\quad yz+y+z=-9,\quad zx+z+x=-17.$

13. $y-yz+z=1-a,\quad z-zx+x=1-b,\quad x-xy+y=1-c.$

14. $(x-y)(y+z)=a^2,\quad (y+z)(z-x)=b^2,\quad (z-x)(x+y)=c^2.$

15. $(x+y-z)(x-y+z)=a^2,\quad (y+z-x)(y-z+x)=b^2,\quad (z+x-y)(z-x+y)=c^2.$

16. $x^2-yz=a^2,\quad y^2-zx=b^2,\quad z^2-xy=c^2.$

17. $x^2+2yz=a^2,\quad y^2+2zx=a^2,\quad z^2+2xy=b^2.$

18. $$x^2+xy+y^2=37,\quad y^2+yz+z^2=19,\quad z^2+zx+x^2=28.$$

19. $$x(y^2-z^2)=-30,\quad y(z^2-x^2)=-12,\quad z(x^2-y^2)=12.$$

20. $$xy(x-y)=12,\quad yz(y-z)=-30,\quad zx(z-x)=48.$$

21. $$xy(x-y)=a^3,\quad yz(y-z)=b^3,\quad zx(z-x)=c^3.$$

22. $$x^2(y-z)=-75,\quad y^2(z-x)=-16,\quad z^2(x-y)=7.$$

23. $$x^2-yz=ax,\quad y^2-zx=by,\quad z^2-xy=cz.$$

24. $$x(x+y+z)-(y^2+z^2+yz)=a^2,$$
$$y(x+y+z)-(z^2+x^2+zx)=b^2,$$
$$z(x+y+z)-(x^2+y^2+xy)=c^2.$$

25. $$(x+y)(x+z)=ax,\quad (y+z)(y+x)=by,\quad (z+x)(z+y)=cz.$$

26. $$(x-y)(y-z)=by,\quad (y-z)(z+x)=cz,\quad (z+x)(x-y)=ax.$$

27. $$x^3+y^3+z^3=3xyz,\quad x-a=y-b=z-c.$$

28. $$6(x^2+y^2+z^2)=\frac{481}{6},\quad 13(x+y+z)=\frac{481}{6},\quad xy=z^2.$$

29. $$x^2(y+z)=a^3,\quad y^2(z+x)=b^3,\quad xyz=c^3.$$

30. $$x+y+z=10,\quad yz+zx+xy=33,\quad (y+z)(z+x)(x+y)=294.$$

31. $$x+y+z=1,\quad x^2+y^2+z^2+6xy=0,\quad \frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=0.$$

32. $$xz=y^2,\quad (x+y)(z-x-y)=3,\quad (x+y+z)(z-x-y)=7.$$

33. $$(x-2)^2+(y-3)^2+(z-1)^2=24,$$
$$xy+yz+zx=63,$$
$$2x+3y+z=30.$$

CHAPTER XIV.

ELIMINATION.

209. From *two* or more equations it is frequently necessary to form an equation in which *one* or more of the quantities previously involved will not appear. The process of finding such an equation is called **Elimination,** and the unknown quantity or quantities which disappear are said to be **Eliminated.**

210. In order that elimination may be effected there must be at least *one independent* equation more than the quantities to be eliminated. The resulting equation is often called an **Equation of Condition.**

Thus from $$ax + by = c \qquad (1)$$

and $$a_1x + b_1y = c_1 \qquad (2)$$

we can obtain the values of x and y, and these values, substituted in

$$a_2x + b_2y = c_2 \qquad (3)$$

will give an equation without x and y, but expressed in terms of the other letters of the equations. This new equation expresses the *condition* that x and y may satisfy the three equations, (1), (2) and (3).

211. It often happens that the student does not at once perceive the number of quantities to be eliminated in the problems given him. Thus, in the following problem:

Eliminate x, y and z from

$$ax + by + cz = 0, \qquad (1)$$

$$a_1x + b_1y + c_1z = 0, \qquad (2)$$

$$a_2x + b_2y + c_2z = 0, \qquad (3)$$

the beginner usually assumes that there are *three* independent quantities to be eliminated; and as there are only *three* equations given, he naturally concludes that elimination is impossible. In such problems, however, there are only *two* independent quantities. For, divide (1), (2) and (3) separately by z, then we obtain,

$$a \cdot \frac{x}{z} + b \cdot \frac{y}{z} + c = 0, \qquad (4)$$

$$a_1 \cdot \frac{x}{z} + b_1 \cdot \frac{y}{z} + c_1 = 0, \qquad (5)$$

$$a_2 \cdot \frac{x}{z} + b_2 \cdot \frac{y}{z} + c_2 = 0. \qquad (6)$$

Thus we see that in (4), (5) and (6) there are only two independent quantities, namely, $\frac{x}{z}$ and $\frac{y}{z}$, and elimination is possible. If, however, (1), (2) and (3) had, on the right-hand side of the sign of equality, quantities such as d, d_1, d_2, elimination would be impossible, as there would now be *three* independent quantities to be eliminated, and only *three* equations.

212. No general rule or rules can be given for Elimination. In simple cases the values of the quantities to be eliminated can be obtained from the same number of equations, and then these values can be substituted in the remaining equation or equations. This process, however, is not always easy or desirable, and various artifices are employed to get rid of the undesired quantities. The following examples may give some assistance:

Ex. 1.—Eliminate x from the equations,

$$ax + b = c \tag{1}$$

and

$$a_1x + b_1 = c_1, \tag{2}$$

From (1),

$$x = \frac{c - b}{a},$$

and from (2),

$$x = \frac{c_1 - b_1}{a_1};$$

$$\therefore \frac{c - b}{a} = \frac{c_1 - b_1}{a_1},$$

which is one form of the desired result.

Ex. 2.—If

$$ax + by = 0 \tag{1}$$

and

$$a_1x + b_1y = 0, \tag{2}$$

eliminate x and y.

From (1),

$$\frac{a}{b} = -\frac{y}{x},$$

and from (2)

$$\frac{a_1}{b_1} = -\frac{y}{x};$$

$$\therefore \frac{a}{b} = \frac{a_1}{b_1}, \text{ or } ab_1 - a_1b = 0.$$

Ex. 3.—If

$$ax + by + cz = 0 \tag{1}$$

and

$$a_1x + b_1y + c_1z = 0, \tag{2}$$

eliminate z, and find the value of $\frac{x}{y}$.

Multiply (1) by c_1 and (2) by c, and subtract the products.

Then $$x(ac_1 - a_1c) + y(bc_1 - b_1c) = 0.$$

Therefore

$$\frac{x}{y} = \frac{b_1c - bc_1}{ac_1 - a_1c}.$$

Ex. 4.—Given

$$\left.\begin{aligned} x+y+z&=a, & (1)\\ xy+yz+zx&=b, & (2)\\ xyz&=c, & (3)\end{aligned}\right\}\text{ eliminate } y \text{ and } z.$$

From (1), $$y+z=a-x,$$

from (2), $$y+z=\frac{b-yz}{x},$$

and from (3), $$yz=\frac{c}{x}.$$

Combining these results we obtain

$$x^3-ax^2+bx-c=0.$$

NOTE.—The student's attention is directed to another and more elegant solution of this problem in the chapter on Simultaneous Equations.

Ex. 5.—Having given $x=by+cz+du$, (1)

$$y=ax+cz+du, \quad (2)$$

$$z=ax+by+du, \quad (3)$$

$$u=ax+by+cz, \quad (4)$$

show that $$1=\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d},$$

x, y, z, u being supposed all unequal.

Adding ax to both sides of (1), by to both sides of (2), cz to (3), and du to (4), we obtain

$$x(1+a)=ax+by+cz+du=y(1+b)=z(1+c)=u(1+d).$$

Let $$ax+by+cz+du=k.$$

Then $$x=\frac{k}{1+a},\quad y=\frac{k}{1+b},\quad z=\frac{k}{1+c},\quad\text{and } u=\frac{k}{1+d}.$$

Substituting the values of x, y, z, u in

$$x(1+a)=ax+by+cz+du,$$

we obtain $$k=\frac{ak}{1+a}+\frac{bk}{1+b}+\frac{ck}{1+c}+\frac{dk}{1+d}.$$

Dividing by k, $$1=\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}.$$

Ex. 6.—Find the relation between a, b, c, having given

$$\frac{y}{z}+\frac{z}{y}=a,\quad \frac{z}{x}+\frac{x}{z}=b,\quad \frac{x}{y}+\frac{y}{x}=c.$$

$$\left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{z}{x}+\frac{x}{z}\right)\left(\frac{x}{y}+\frac{y}{x}\right)=abc,$$

that is, $$\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x^2}{z^2}+\frac{z^2}{x^2}+\frac{y^2}{z^2}+\frac{z^2}{y^2}+2=abc.$$

But $$a^2+b^2+c^2=\left(\frac{y}{z}+\frac{z}{y}\right)^2+\left(\frac{z}{x}+\frac{x}{z}\right)^2+\left(\frac{x}{y}+\frac{y}{x}\right)^2$$

$$=\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x^2}{z^2}+\frac{z^2}{x^2}+\frac{y^2}{z^2}+\frac{z^2}{y^2}+6$$

$$=abc+4.$$

$$\therefore\ a^2+b^2+c^2-4=abc,$$

or $$a^2+b^2+c^2-abc=4.$$

Ex. 7.—Eliminate x from the equations,

$$ax^3+bx+c=0, \tag{1}$$

$$a_1x^3+b_1x+c_1=0, \tag{2}$$

As in Art. 39, Exs. 1 and 2, we have

$$\frac{x^3}{bc_1-b_1c}=\frac{x}{ca_1-c_1a}=\frac{1}{ab_1-a_1b}.$$

$$\therefore\ x^3=\frac{bc_1-b_1c}{ab_1-a_1b}\ \text{ and }\ x=\frac{ca_1-c_1a}{ab_1-a_1b};$$

$$\therefore\ \left(\frac{ca_1-c_1a}{ab_1-a_1b}\right)^3=\frac{bc_1-b_1c}{ab_1-a_1b},$$

or $$(ab_1-a_1b)^2(bc_1-b_1c)=(ca_1-c_1a)^3.$$

Ex. 8.—Eliminate x, y, z from

$$(x-y+z)(y-z+x)=ayz, \qquad (1)$$
$$(y-z+x)(z-x+y)=bzx, \qquad (2)$$
$$(z-x+y)(x-y+z)=cxy. \qquad (3)$$

Multiplying together (1), (2) and (3),

$$\{(x-y+z)(y-z+x)(z-x+y)\}^2=abcx^2y^2z^2,$$

or
$$\frac{(x-y+z)(y-z+x)(z-x+y)}{xyz}=\sqrt{abc},$$

or
$$\frac{x^2y+x^2z+y^2z+y^2x+z^2x+z^2y-x^3-y^3-z^3-2xyz}{xyz}=\sqrt{abc},$$

or
$$\frac{4xyz-x\{x^2-(y-z)^2\}-y\{y^2-(z-x)^2\}-z\{z^2-(x-y)^2\}}{xyz}=\sqrt{abc},$$

or
$$\frac{4xyz-axyz-bxyz-cxyz}{xyz}=\sqrt{abc},$$

or
$$4-a-b-c=\sqrt{abc}.$$
$$\therefore\ (4-a-b-c)^2=abc.$$

EXERCISE XX.

1. Eliminate x and y from $x+y=z$, $x^2+y^2=a^2$, $x^3+y^3=b^3$.

2. Eliminate x and y from $x+y=a$, $xy=z^2$, $x^7+y^7=b^7$.

3. Eliminate x, y and z from the equations,
$$x^2(y+z)=a^3,\ y^2(x+z)=b^3,\ z^2(x+y)=c^3,\ xyz=abc.$$

4. Eliminate m, n, p, q from the equations,
$$\frac{x-p}{m}+\frac{y+q}{n}=\frac{pm}{a^2}+\frac{qn}{b^2}=\frac{m^2}{a^2}-\frac{n^2}{b^2}=\frac{p^2}{a^2}+\frac{q^2}{b^2}-1=0.$$

5. Eliminate x and y from the equations,
$$ax+by=x+y+xy=x^2+y^2-1=0.$$

6. Given $y^2 - x^2 = ay - bx$, $4xy = ax + by$ and $x^2 + y^2 = 1$, eliminate x and y, and show that $(a+b)^{\frac{2}{3}} + (a-b)^{\frac{2}{3}} = 2$.

7. Eliminate x and y from the equations,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a, \quad b = x + 3x^{\frac{1}{3}}y^{\frac{2}{3}}, \quad c = y + 3x^{\frac{2}{3}}y^{\frac{1}{3}}.$$

8. Find the condition that

$$a_1x + b_1y = c_1, \quad a_2x + b_2y = c_2 \text{ and } a_3x + b_3y = c_3$$

may be satisfied by the same values of x and y.

9. Eliminate x, y and z from the equations,

$$(b+c)x + (c+a)y + (a+b)z = 0, \quad (b-c)x + (c-a)y + (a-b)z = 0$$
$$\text{and } x^{-1} + y^{-1} + z^{-1} = 0.$$

10. Eliminate x, y, z from

$$ax + by + cz = bx + cy + az = cx + ay + bz = 1 \text{ when } x^2 + y^2 + z^2 = p^2.$$

11. Eliminate x, y, z from

$$bx + cy + az = cx + ay + bz = ax + by + cz = ab + bc + ca$$
$$\text{and } x + y + z = a + b + c.$$

12. If $ax + c_1y + b_1z = 0$, $c_1x + by + a_1z = 0$, $b_1x + a_1y + cz = 0$,

show that $$aa_1^2 + bb_1^2 + cc_1^2 = abc + 2a_1b_1c_1$$

and $(ab - c_1^2)(ca - b_1^2)x^2 = (bc - a_1^2)(ab - c_1^2)y^2 = (bc - a_1^2)(ac - b_1^2)z^2$.

13. Eliminate u, x, y, z from the equations,

$$y + z + u = ax, \quad z + u + x = by, \quad u + x + y = cz, \quad x + y + z = du.$$

14. Eliminate a and b from the equations,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x}{a} + \frac{y}{b} = 1, \quad \frac{h}{a} + \frac{k}{b} = 1.$$

15. Eliminate x or y from the equations,

$$x^3 + 3x^2y + 3xy^2 + 2y^3 = 0, \quad x^2 + xy + y^2 = 0.$$

16. Eliminate x or y from

$$(x-y)(x^2-y^2)=5 \text{ and } (x+y)(x^2+y^2)=65.$$

17. Eliminate x or y from

$$x^2+y^3+(2y^2+1)(x+1)=0 \text{ and } x^2-y^2-xy-1=0.$$

18. Eliminate x and y from

$$ax+by=c^2, \quad xy=c^2-ab \text{ and } \frac{x^3}{a^3}+\frac{y^3}{b^3}=\frac{1}{8}.$$

19. Eliminate x and y from

$$ax^2+bxy+cy^2=0 \text{ and } a_1x^2+b_1xy+c_1y^2=0.$$

20. Eliminate x and y from

$$x^2+xy+y^2=a^2, \quad x^4+x^2y^2+y^4=b^4 \text{ and } x^8+x^4y^4+y^8=c^8.$$

21. Eliminate x and y from

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{b^2}-\frac{y^2}{a^2}=0 \text{ and } \frac{x^4}{a^4}-\frac{y^4}{b^4}=\frac{m}{n}.$$

22. Eliminate x and y from

$$a^2by=mb^2x, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } bx+my=p.$$

23. Eliminate x and y from

$$y^2(bx-ay)=x^2(ax-by)=x^2y^2 \text{ and } x^2+y^2=c^2.$$

24. Eliminate x and y from

$$x+\frac{ab}{x}=2m, \quad y+\frac{ac}{y}=2n \text{ and } \frac{cx}{y}+\frac{by}{x}=p.$$

25. Eliminate x and y from

$$ax+by=c(x^2+y^2)^{\frac{1}{2}} \text{ and } a_1x+b_1y=c_1(x^2+y^2)^{\frac{1}{2}}.$$

26. If $y^2-m(2x+m)=a^2$, $x^2-m(2y+m)=b^2$ and $a+x=b+y$, show that $m=0$ or $m=a+b$.

27. Find the equation between x, y and z, independent of a and b, from the equations, $a^3+ax=y$, $b^3+bx=z$ and $a^2+b^2=1$.

28. From $ax+by-z=(m^2a^2+n^2b^2+c^2)^{\frac{1}{2}}=n^2by^{-1}=m^2ax^{-1}$ find an equation which does not contain a and b.

29. Given the equations, $x^3=\dfrac{1-z}{1+z}$ and $y=\dfrac{2z}{1-z^2}$, show that if z be eliminated the following equation will hold between x and y:

$$\{y+(y^2+1)^{\frac{1}{2}}\}^{\frac{1}{3}}+\{y-(y^2+1)^{\frac{1}{2}}\}^{\frac{1}{3}}=x^{-1}-x.$$

30. Eliminate y and z from the equations,

$$y+z=a,\quad x^2+y^2-2mxy=b^2 \text{ and } x^2+z^2+2mxz=c^2,$$

and show that $\dfrac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4a^2(1-m^2)}=x^2.$

31. Eliminate a from the equations, $\dfrac{x}{a^2+x^2}=\dfrac{2y}{a^2+y^2}=\dfrac{4z}{a^2+z^2}$, and prove that $x(y^2-z^2)+2y(z^2-x^2)+4z(x^2-y^2)=0$.

32. If $ax^2+by^2=bz(y-x)$ and $bx^2+ay^2=ay(x-z)$, find the relation between x, y and z.

33. Eliminate x, y, and z from

$$x+y+z=a,\quad x^2+y^2+z^2=b^2 \text{ and } x^3+y^3+z^3-3xyz=c^3.$$

34. Eliminate x, y and z from

$$x+y+z=a,\quad x^2+y^2+z^2=b^2,\quad x^3+y^3+z^3=c^3 \text{ and } xyz=d^3.$$

35. Eliminate x, y and z from

$$x+y+z=0 \text{ and } \frac{x}{a}+\frac{a}{x}=\frac{b}{y}+\frac{y}{b}=\frac{c}{z}+\frac{z}{c}.$$

36. Eliminate x, y and z from

$$xyz=a^3,\ (x+y)(y+z)(z+x)=c^3,\ \frac{x}{y}+\frac{y}{z}+\frac{z}{x}=a \text{ and } \frac{x}{z}+\frac{y}{x}+\frac{z}{y}=b.$$

37. Eliminate x, y and z from

$$x^2=y^2+z^2-2ayz,\quad y^2=x^2+z^2-2bzx \text{ and } z^2=x^2+y^2-2cxy.$$

38. Eliminate x, y and z from

$$x^2(y+z)=a^3,\ y^2(z+x)=b^3,\ z^2(x+y)=c^3,\ (x+y)(y+z)(z+x)=abc.$$

39. Eliminate x, y and z from

$$(x+y)^2=4cxy,\ (x+z)^2=4bxz \text{ and } (y+z)^2=4ayz.$$

40. Eliminate x, y and z from

$$x^2-yz=a,\ y^2-zx=b,\ z^2-xy=c,\ ax+by+cz=d.$$

41. Eliminate x, y and z from

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2+k^2}+\frac{y^2}{b^2+k^2}+\frac{z^2}{c^2+k^2}=1 \text{ and } \frac{x}{a^2x_1}=\frac{y}{b^2y_1}=\frac{z}{c^2z_1}.$$

42. If $\dfrac{x}{y+z}=a$, $\dfrac{y}{z+x}=b$ and $\dfrac{z}{x+y}=c$, find the relation between a, b and c.

43. Find the relation between a, b and c, having given

$$\frac{x}{a}+\frac{a}{x}=\frac{y}{b}+\frac{b}{y}=\frac{z}{c}+\frac{c}{z},\ xyz=abc,\ x^2+y^2+z^2+2(ab+bc+ca)=0.$$

44. Eliminate a and b from the equations,

$$\frac{a^3-x^3}{b^3-y^3}=\frac{2x+3y}{3x+2y},\ a^3-b^3=(x-y)^3 \text{ and } a^{\frac{3}{2}}+b^{\frac{3}{2}}=z^{\frac{3}{2}}.$$

45. Eliminate x and y from $x+y=a$, $x^3+y^3=b^3$ and $x^5+y^5=c^5$.

46. Eliminate x from the equations,

$$32\frac{c}{a}=\left(\frac{x}{a}\right)^5+10\left(\frac{x}{a}\right)+5\left(\frac{a}{x}\right)^3,\ 32\frac{a}{c}=\left(\frac{a}{x}\right)^5+10\left(\frac{a}{x}\right)+5\left(\frac{x}{a}\right)^3.$$

47. Eliminate x, y and z from the equations,

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=a,\ \frac{x}{z}+\frac{y}{x}+\frac{z}{y}=b,\ \left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)=c.$$

48. Eliminate x and y from

$$4(x^2+y^2)=ax+by,\ 2(x^2-y^2)=ax-by \text{ and } xy=c^2.$$

CHAPTER XV.

THEORY OF QUADRATICS.

213. In Part I., Chapter XXI., the simpler portion of the Theory of Quadratics has been treated. It remains now to give that part of the Theory which is of a more difficult nature.

214. In algebraical researches it is frequently necessary to determine for what values of the variable a particular quadratic expression becomes positive or negative. A few numerical illustrations will render the principal proposition more easily intelligible.

Take for example the expression, $x^2 - 13x + 36$, and for x substitute various numbers, and collect the result. Thus when

$$x = \ 1, \ \ 2, \ 3, \ 4, \ \ 5, \ \ 6, \ \ 7, \ \ 8, \ 9, \ 10, \ 11 \ldots.$$
$$x^2 - 13x + 36 = 24, \ 14, \ 6, \ 0, \ -4, \ -6, \ -6, \ -4, \ 0, \ 6, \ 14 \ldots.$$

It will be observed that when x is 4 or 9 the expression vanishes, because these are the roots of the equation,

$$x^2 - 13x + 36 = 0;$$

that when x is *between* those roots the expression is *negative*, and that for all other values it is positive. If we change the sign of every term in the expression, the vanishing points are unchanged; but the expression is now positive when x is between 4 and 9, and negative for all other values.

Next consider the expression, $2x^2 - 8x + 25$. It will be found that this expression is always positive for real values of x. When $x = 2$ the expression becomes 17; but for all other real values of

x it is greater. The reason for this peculiarity is seen at once by writing the expression in this form:

$$2\{(x-2)^2+8\tfrac{1}{2}\}.$$

When $x=2$, $(x-2)^2$ is zero; but for all other real values of x it is a positive quantity. Note that in this case *the roots of*

$$2x^2-8x+25=0$$

are imaginary.

We will now proceed to state and prove the general proposition relating to this subject.

215. *The quadratic expression, ax^2+bx+c, is always of the same sign as a for all real values of x, except when the roots of the quadratic equation, $ax^2+bx+c=0$, are* REAL *and* UNEQUAL, *and x lies between them.*

Let $$ax^2+bx+c=a(x-m)(x-n).$$

Then the roots of $ax^2+bx+c=0$ are m and n. (Art. 294, Pt. I.)

1. Let $m=n$. Then $ax^2+bx+c=a(x-n)^2$. But $(x-n)^2$ is positive when x and n are real; $\therefore$ $a(x-n)^2$ is positive if a is positive, and negative if a is negative.

2. Let $m>n$. Then if $x>m$ and $>n$, $(x-m)(x-n)$ is positive, since both factors are positive. Also, if $x<m$ and $<n$, $(x-m)(x-n)$ is positive, since both factors are negative. But if $x>m$ and $<n$, or $x<m$ and $>n$, then $(x-m)(x-n)$ is negative, since one factor is positive and the other negative. When $(x-m)(x-n)$ is positive, $a(x-m)(x-n)$, as shown before, is of the same sign as a; but when $(x-m)(x-n)$ is negative, then $a(x-m)(x-n)$ must have a different sign from a.

3. Let the roots of $ax^2+bx+c=0$ be imaginary.

Solving $$ax^2+bx+c=0$$

we find $$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

If $b^2<4ac$, the roots must be imaginary.

Now, $$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$
$$=a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}\right)$$
$$=a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\}.$$

But $\left(x+\dfrac{b}{2a}\right)^2$ is positive; and $\dfrac{4ac-b^2}{4a^2}$ is positive, since $b^2<4ac$ and $4a^2$ is positive. $\therefore$ ax^2+bx+c equals the product of a and a positive quantity; $\therefore$ ax^2+bx+c must be of the same sign as a when the roots of ax^2+bx+c are imaginary.

Ex. 1.—The expression, $2x^2+8x+9$, is always positive for all real values of x.

Since $8^2<4\times 2\times 9$, the roots of $2x^2+8x+9=0$ are imaginary; $\therefore$ the value of $2x^2+8x+9$ for all real values of x is of the same sign as 2, and is therefore positive.

Ex. 2.—The expression, $4x^2+15x+12$, is always positive except for values of x lying between the roots of $4x^2+15x+12=0$. For $15^2>4\times 4\times 12$; $\therefore$ the roots of $4x^2+15x+12$ are real and unequal, and, therefore, $4x^2+15x+12$ is negative when x lies between these roots.

216. It is sometimes possible to find maximum and minimum values of a fraction for real values of the unknown quantity.

Let it be required to find the maximum and minimum values of
$$\frac{x^2-3x-3}{2x^2+2x+1}.$$

Assume $$\frac{x^2-3x-3}{2x^2+2x+1}=k.$$

Multiplying out, and arranging as a quadratic,
$$x^2(1-2k)-x(3+2k)-(3+k)=0. \qquad (1)$$

Since x is real, $(3+2k)^2+4(1-2k)(3+k) \geqq 0$,

or $$21-8k-4k^2 \geqq 0,$$

or $$4k^2+8k-21 \leqq 0,$$

or $$(2k+7)(2k-3) \leqq 0.$$

Now, in order to be negative or < 0, $2k$ must be < 3, and $\therefore k < \frac{3}{2}$, so that the maximum value of k is $\frac{3}{2}$. Also, if $2k < 3$, $2k+7$ must be positive; $\therefore 2k > -7$, and $\therefore k > -\frac{7}{2}$. The minimum value of k is therefore $-\frac{7}{2}$, and the maximum value, $\frac{3}{2}$. If $k = -\frac{7}{2}$ or $\frac{3}{2}$, the expression, $(2k+7)(2k-3)=0$, which is the condition that the roots of (1) should be real and equal.

217. *To find the condition that* $ax^2+2hxy+by^2+2gx+2fy+c$ *may resolve into two rational factors, the constant quantities,* a, h, b, g, f, c, *being rational.*

The expression, $ax^2+2hxy+by^2+2gx+2fy+c$, will resolve into two rational factors if the equation,

$$ax^2+2hxy+by^2+2gx+2fy+c=0, \tag{1}$$

when arranged as a quadratic in x or y, has rational roots.

Arrange (1) as follows:

$$ax^2+x(2hy+2g)+(by^2+2fy+c)=0.$$

When the equation is solved the quantity under the radical sign will be

$$(2hy+2g)^2-4a(by^2+2fy+c). \tag{2}$$

In order that the roots may be rational (2) must be a perfect square.

Now, $$(2)=4\{y^2(h^2-ab)+y(2gh-2af)+(g^2-ac)\},$$

the condition of which being a perfect square is

$$(2gh-2af)^2=4(h^2-ab)(g^2-ac),$$

or $$4(gh-af)^2=4(h^2-ab)(g^2-ac),$$

or $$(gh-af)^2=(h^2-ab)(g^2-ac).$$

218. To find the condition that $ax^2+bx+c=0$ may have roots equal numerically, but of different signs.

Since $$ax^2+bx+c=0,$$

then $$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

The sum of the roots of this equation is $-\dfrac{b}{a}$.

But since the roots are equal numerically, and of different signs, the sum of roots $=0$.

$$\therefore \quad -\frac{b}{a}=0 \text{ or } b=0 \text{ is the condition required.}$$

EXERCISE XXI.

1. Resolve $2y^2+2x^2-5xy-4ay-ax-6a^2$ into two simple factors.

2. If $y^2+axy+bx^2+cy+dx+e$ can be resolved into two rational factors of the first degree, find the relation between the coefficients.

3. If $ax^2+by^2+2a_1y+2b_1x+2c_1xy+c$ be the product of two rational lineal factors, show that

$$abc+2a_1b_1c_1-aa_1^2-bb_1^2-cc_1^2=0.$$

4. If x be real, $\dfrac{9}{x-1}-\dfrac{4}{x-6}$ can never lie between 5 and $\dfrac{1}{5}$.

5. If x be real, prove that $\dfrac{x^2+5x+4}{x^2+5x}$ can have no real value between $\dfrac{9}{25}$ and 1.

6. If n be real, prove $\dfrac{n^2-n+1}{n^2+n+1}$ must lie between 3 and $\dfrac{1}{3}$.

7. The greatest value which $\dfrac{a(x+a)}{x^2+a^2}$ admits of for any real value of x is $\dfrac{\sqrt{2}+1}{2}$.

8. Show that for real values of x, $\dfrac{x^2-4x+4}{x-1}$ cannot lie between 0 and -4.

9. Within what limits is the value of $\dfrac{x^2-3x-3}{2x^2+2x+1}$ an integer or a rational fraction?

10. Show that $(x-a)(b-x)$ can never exceed $\dfrac{1}{4}(a-b)^2$.

11. Show that the least value of $\dfrac{(x+a)(x+b)}{x}$ is $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2$, and the greatest value of $\dfrac{(x+a)(x-b)}{x^2}$ is $\dfrac{(a+b)^2}{4ab}$.

12. Show that the roots of $(3a-x)^{-1}+(3b-x)^{-1}+(3c-x)^{-1}=0$ are real if a, b, c are real.

13. Show that by giving an appropriate real value to x,

$$\frac{4x^2+36x+9}{12x^2+8x+1}$$

can be made to assume any real value.

14. The expression $\dfrac{(x-a)(x-b)}{(x-c)(x-d)}$, admits of all possible values, provided that *one*, and *only* one, of the quantities a or b lies between c and d, and otherwise will have two limits between which it cannot lie.

15. The expression, $\dfrac{ax^2+2bx+c}{a_1x^2+2b_1x+c_1}$, will be capable of all values for real values of x, provided that $(ac_1-a_1c)^2<4(a_1b-ab_1)(b_1c-bc_1)$.

16. If a be $>b$, and c be positive, prove that the greatest value which the expression, $(x-a)(x-b)(x-a-c)(x-b+c)$, can have for values of x between a and b is

$$\frac{(a-b)^2(a-b+2c)^2}{16}.$$

17. Find the limits between which a must lie in order that

$$\frac{ax^2-7x+5}{5x^2-7x+a}$$

may be capable of all values, x being any real quantity.

18. Show that $\dfrac{px^2+3x-4}{p+3x-4x^2}$ will be capable of all values when x is real, provided that p has any value between 1 and 7.

19. If the roots of $ax^2+2bx+c=0$ be possible and different, then the roots of $(a+c)(ax^2+2bx+c)=2(ac-b^2)(x^2+1)$ will be impossible, and *vice versâ*.

20. Show that $\dfrac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$ will be capable of all values when x is real, if (c^2-d^2) and (a^2-b^2) have the same sign.

21. For what values of m will the expression,

$$y^2+2xy+2x+my-3,$$

be capable of resolution into two rational factors?

22. Find the values of m which will make

$$2x^2+mxy+3y^2-5y-2$$

equivalent to the product of two linear factors.

23. Show that $m(x^2-y^2)-xy(l-n)$ always admits of two real linear factors.

24. Find the condition that

$$lx^2 + mxy + ny^2 \text{ and } l_1x^2 + m_1xy + n_1y^2$$

may have a common linear factor.

25. If the expression, $3x^2 + 2Axy + 2y^2 + 2ax - 4y + 1 = 0$, can be resolved into linear factors, prove that A must be one of the roots of the equation, $A^2 + 4aA + 2a^2 + 6 = 0$.

26. Find the condition that the expressions,

$$ax^2 + 2hxy + by^2 \text{ and } a_1x^2 + 2h_1xy + b_1y^2,$$

may be respectively divisible by factors of the form $y - mx$ and $my + x$.

27. Show that in the equation,

$$x^2 - 3xy + 2y^2 - 2x - 3y - 35 = 0,$$

for every real value of x there is a real value of y, and for every real value of y there is a real value of x.

28. If x and y are two real quantities connected by the equation, $9x^2 + 2xy + y^2 - 92x - 20y + 244 = 0$, then will x lie between 3 and 6, and y between 1 and 10.

29. If $(ax^2 + bx + c)y + a_1x^2 + b_1x + c_1 = 0$, find the condition that x may be a rational function of y.

30. The expression, $\dfrac{ax^2 + bx + c}{cx^2 + bx + a}$, will be capable of all values whatever if $b^2 > (a + c)^2$. There will be two values between which it cannot lie if $b^2 < (a + c)^2$ and $> 4ac$, and two values between which it must lie if $b^2 < 4ac$.

31. Find the greatest numerical value, without regard to sign, which the expression, $(x - 8)(x - 14)(x - 16)(x - 22)$, can have for values of x between 8 and 22.

SOME RELATIONS BETWEEN ROOTS AND THEIR COEFFICIENTS.

219. Theorem.—*If the coefficients of an equation are rational, then surd roots must be in pairs; i.e., if $a+\sqrt{b}$ is one root, then $a-\sqrt{b}$ must be another root.*

Ex.—Let $a+\sqrt{b}$ be a root of $x^3+px^2+qx+r=0$.

Then, substituting $a+\sqrt{b}$ for x we obtain

$$(a+\sqrt{b})^3+p(a+\sqrt{b})^2+q(a+\sqrt{b})+r=0,$$

or $$\{a^3+3ab+pa^2+pb+qa+r\}+\{3a^2+b+2pa+q\}\sqrt{b}=0.$$

But when the sum of a rational quantity and a surd is equal to zero, the rational quantity must be equal to zero; also the surd equal to zero.

$$\therefore\ a^3+3ab+pa^2+pb+qa+r=0, \qquad (1)$$

and $$(3a^2+b+2pa+q)\sqrt{b}=0, \qquad (2)$$

Subtracting (2) from (1),

$$a^3-3a^2\sqrt{b}+3ab-b\sqrt{b}+p(a^2-2a\sqrt{b}+b)+q(a-\sqrt{b})+r=0,$$

or $$(a-\sqrt{b})^3+p(a-\sqrt{b})^2+q(a-\sqrt{b})+r=0.$$

From this it is evident that $(a-\sqrt{b})$ is a root of

$$x^3+px^2+qx+r=0.$$

The following is the

General Proof.—Let the equation,

$$x^n+px^{n-1}+qx^{n-2}+\ldots.$$

have one root, $a+\sqrt{b}$, and let the coefficients, p, q, $r\ldots.$ be rational.

If $a+\sqrt{b}$ is substituted for x the equation will take the form of

$$P+Q\sqrt{b}=0,$$

where P stands for the rational terms and $Q\sqrt{b}$ the irrational.

Since $$P+Q\sqrt{b}=0,$$

$$\therefore\ P=0 \text{ and } Q\sqrt{b}=0.$$

$$\therefore\ P-Q\sqrt{b}=0.$$

But if $a-\sqrt{b}$ be substituted for x in the equation the result obtained will be $P-Q\sqrt{b}$; and as $P-Q\sqrt{b}=0$, $a-\sqrt{b}$ must be a root of the equation. See Art. 186 for a shorter proof.

220.—Theorem.—*If the coefficients of an equation are* REAL, *then if $a+b\sqrt{-1}$ is a root, $a-b\sqrt{-1}$ is also a root; or, in brief, if the coefficients of an equation are real, imaginary roots occur in pairs.*

The proof of this proposition can be obtained by the same line of reasoning as in the preceding theorem.

Ex.—Let $a+b\sqrt{-1}$ be a root of $x^2+px+q=0$.

Then $$(a+b\sqrt{-1})^2+p(a+b\sqrt{-1})+q=0,$$

or $$(a^2-b^2+pa+q)+(2ab+pb)\sqrt{-1}=0,$$

or $$a^2-b^2+pa+q=0$$

and $$(2ab+pb)\sqrt{-1}=0.$$

Subtracting, $$a^2-2ab\sqrt{-1}-b^2+p(a-b\sqrt{-1})+q=0,$$

i.e., $$(a-b\sqrt{-1})^2+p(a-b\sqrt{-1})+q=0.$$

From this condition it is evident that $a-b\sqrt{-1}$ is a root of $x^2+px+q=0$.

EXERCISE XXII.

Solve the equations:

1. $3x^4 - 10x^3 + 4x^2 - x - 6 = 0$ when one root is $\dfrac{1 + \sqrt{-3}}{2}$.

2. $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$ when one root is $2 - \sqrt{3}$.

3. $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$ when one root is $-1 + \sqrt{-1}$.

4. $x^5 - x^4 + 8x^2 - 9x - 15 = 0$, one root being $\sqrt{3}$ and another $1 - 2\sqrt{-1}$.

5. Form the equation of the lowest dimensions with rational coefficients, one of whose roots is:

(1) $\sqrt{3} + \sqrt{-2}$; (2) $-\sqrt{-1} + \sqrt{5}$;

(3) $-\sqrt{2} - \sqrt{-1}$; (4) $\sqrt{5 + 2\sqrt{6}}$.

6. Form the equation whose roots are $\pm 4\sqrt{3}$, $5 \pm 2\sqrt{-1}$.

7. Form the equation of the eighth degree with rational coefficients, one of whose roots is $\sqrt{2} + \sqrt{3} + \sqrt{-1}$.

8. If $\alpha \pm \beta\sqrt{-1}$ be the roots of $x^3 + qx + r = 0$, then $\beta^2 = 3\alpha^2 + q$.

9. If $x^3 + px^2 + qx + r = 0$ be satisfied by $x = 3 + \sqrt{8}$, it will be satisfied by $x = -r$.

10. If $\dfrac{a}{2} + \sqrt{b}$ be a root of $x^3 + px^2 + qx + r = 0$, then $a^2 + pa + q$ is a factor of r, b not being a perfect square, and p, q, r rational.

11. If $a + b\sqrt{-1}$ be a root of $x^3 + qx + r = 0$, then a is a root of $8x^3 + 2qx - r = 0$ and $3a^2 - b^2 = -q$; but if $a + b\sqrt{-1}$ be a root of $x^3 - px^2 - r = 0$, then a is a root of $8x^2 - 8px^2 + 2p^2x + r = 0$.

12. If $x^3 + qx + r = 0$ have a root $\dfrac{1}{2}(a + \sqrt{b})$, show that a is a root of the equation, $x^3 + qx - r = 0$.

CHAPTER XVI.

INDETERMINATE COEFFICIENTS.

221. The student of Algebra is frequently called upon to determine the relations between the roots of an equation and its coefficients; also the conditions that must be satisfied by the coefficients of an algebraic expression when it vanishes for certain definite values of the unknown quantity involved.

Ex.—What condition must be satisfied by the coefficients of ax^2+bx+c if this expression vanishes for more than two given values of x ax^2+bx+c being a positive integral function of x?

Let ax^2+bx+c vanish when $x=m$, n, r. Then

$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a(x-m)(x-n)(x-p); \qquad \text{Part I., Art. 275}$$

therefore

$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=ax^3-ax^2(m+n+p)+ax(mn+np+pm)-amnp,$$

a quantity of the third degree, which is impossible unless

$$ax^3=0 \text{ or } a=0.$$

If $a=0$, then $\qquad ax^2+bx+c=bx+c.$

In the same way it can be shown that $b=0$, and $\therefore$ $c=0$.

The conditions, then, for ax^2+bx+c vanishing for more than *two* values of x are $a=0$, $b=0$, $c=0$.

222. We proceed now to give the proof of the general proposition of which the preceding example is an illustration:

Proposition.—*If any positive integral function of x of the n^{th} degree vanish for more than n different values of x, then each of the coefficients must vanish.*

Let $A_nx^n + A_{n-1}x^{n-1} + \ldots A_0$ be a positive integral function of x. This function can be denoted by the symbol $f(x)$.

Let $a_1, a_2 \ldots . a_n$ be values of x which make $f(x)$ vanish.

Then $$f(x) = A_n(x - a_1)(x - a_2)(x - a_3) \ldots . (x - a_n).$$

If possible let $f(x)$ vanish for $x = a_{n+1}$.

Then $$f(a_{n+1}) = 0 = A_n(a_{n+1} - a_1)(a_{n+1} - a_2) \ldots . (a_{n+1} - a_n).$$

Now, since $a_{n+1} - a_1$, $a_{n+1} - a_2 \ldots . a_{n+1} - a_n$ are each not zero, since a_{n+1} is different from $a_1, a_2 \ldots . a_n$, $\therefore$ A_n must $= 0$.

Therefore $f(x)$ reduces to $A_{n-1}x^{n-1} + \ldots . A_0$. In the same manner $A_{n-1}, A_{n-2} \ldots A_0$ can be shown to be zero.

The above is the usual proof; but it is well for the student to recognize that the truth of the proposition depends upon the fact that a positive integral function of x cannot have more linear factors than is indicated by the highest power of x. If it has, then the coefficients of the function must be separately zero.

223. Corollary.—*If $A + Bx + Cx^2 + \ldots . Nx^n = A_1 + B_1x + C_1x^2 + \ldots . N_1x^n$ for more than n values of x, then $A = A_1$, $B = B_1$, $C = C_1 \ldots . N = N_1$.*

For, transposing, $(A - A_1) + (B - B_1)x + \ldots . (N - N_1)x^n = 0$, and the equation is satisfied by more than n different values of x. Therefore the coefficients separately vanish; *i.e.*,

$$A - A_1 = 0, \; B - B_1 = 0 \; \ldots . \; N - N_1 = 0,$$
$$A = A_1, \; B = B_1, \; C = C_1 \; \ldots . \; N = N_1.$$

Ex. 1.—Let $x^3 - px^2 + qx - r = (x-2)(x-3)(x-4)$ for more than two values of x.

Then, since $\quad (x-2)(x-3)(x-4) = x^3 - 9x^2 + 26x - 24,$

we find, by equating coefficients, that $p = 9$, $q = 26$, $r = 24$.

Ex. 2.—Let $ax^4+px^3+qx^2+rx+s=4x^4+3x^2+5x-2$ for more than four values of x.

Then $$a=4,\quad p=0,\quad q=3,\quad r=5,\quad s=-2.$$

This principle is called that of **Indeterminate Coefficients,** by which is meant "coefficients that have to be determined." As the principle has a very extensive application we append a few more solved examples.

Ex. 3.—Show that the following is an identity:

$$\frac{a^2(x-b)(x-c)}{(a-b)(a-c)}+\frac{b^2(x-c)(x-a)}{(b-c)(b-a)}+\frac{c^2(x-a)(x-b)}{(c-a)(c-b)}=x^2.$$

From inspection we see that the equality holds when for x we substitute a, b or c. But the expression is of the second degree in x; and since it is satisfied by more than *two* different values of x, the coefficients of corresponding powers of x on opposite sides of the equation must be equal. Therefore the equation is an identity.

Ex. 4.—Find values for a and b which render the fraction,

$$\frac{2x^2+(a-b)x+2a^2-3b^2}{3x^2+(a-7)x+3(a^2+2ab+3b^2)},$$

the same for all values of x.

Let $$\frac{2x^2+(a-b)x+2a^2-3b^2}{3x^2+(a-7)x+3(a^2+2ab+3b^2)}=k$$

where k is the same for all values of x.

Clearing of fractions, and arranging as a quadratic in x, we obtain

$$x^2(2-3k)+x\{a-b-k(a-7)\}+2a^2-3b^2-3k(a^2+2ab+3b^2)=0.$$

Now, since the left-hand expression vanishes for all values of x, the coefficients separately vanish.

Therefore (1) $2-3k=0$, (2) $a-b-k(a-7)=0$,

(3) $2a^2-3b^2-3k(a^2+2ab+3b^2)=0$.

From these equations we find that $k=\dfrac{2}{3}$, $a=-6$ or -14, $b=2\frac{2}{3}$ or 0.

Ex. 5.—Resolve $2x^2-21xy-11xy^2-x+34y-3$ into rational factors of the first degree.

Assume $2x^2-21xy-11y^2-x+34y-3=(2x+my+n)(x+ry+s)$.

Multiplying out, and equating coefficients of like terms, we have

$$-21=2r+m, \tag{1}$$
$$-11=mr, \tag{2}$$
$$-1=n+2, \tag{3}$$
$$34=nr+ms, \tag{4}$$
$$-3=ns, \tag{5}$$

Solving these equations we find $m=1$, $n=-3$, $r=-11$, and $s=1$.

$\therefore$ factors are $(2x+y-3)(x-11y+1)$.

This is a tedious process of obtaining the factors of this expression, but it is here introduced to illustrate a method sometimes useful.

Ex. 6.—If a, b, c, d are the roots of $x^4-px^3+qx^2-rx+s=0$, find the relations between the roots and the coefficients.

If a, b, c, d are values of x that make $x^4-px^3+qx^2-rx+s$ vanish, then $x-a$, $x-b$, $x-c$ and $x-d$ are factors of this expression, and therefore

$$x^4-px^3+qx^2-rx+s=(x-a)(x-b)(x-c)(x-d)$$

for all values of x.

Multiplying out, and equating coefficients, we obtain

$$a+b+c+d=p, \tag{1}$$

$$ab+bc+cd+da+bd+ca=q, \tag{2}$$

$$abc+bcd+cda+adb=r, \tag{3}$$

$$abcd=s. \tag{4}$$

(1), (2), (3) and (4) are the relations required. From this example it is easily seen what general relations exist between the roots and coefficients of *positive integral* equations.

Ex. 7.—If $\dfrac{x^2}{(x-a)(x-b)(x-c)}=\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$, (1)

find the values of A, B and C, and prove that

$$\frac{a}{(a-b)(a-c)}+\frac{b}{(b-a)(b-c)}+\frac{c}{(c-a)(c-b)}=0.$$

Clearing (1) of fractions we obtain

$$x^2=A(x-b)(x-c)+B(x-a)(x-c)+C(x-a)(x-b). \tag{2}$$

Since (2) is an identity the equality holds for all values of x.

Therefore when $x=a$, $a^2=A(a-b)(a-c)$, or $A=\dfrac{a^2}{(a-b)(a-c)}$. (3)

Also when $x=b$, $b^2=B(b-a)(b-c)$, or $B=\dfrac{b^2}{(b-a)(b-c)}$. (4)

Similarly when $x=c$, $c^2=C(c-a)(c-b)$, or $C=\dfrac{c^2}{(c-a)(c-b)}$. (5)

Again, in (1) let $x=0$; then $\dfrac{A}{a}+\dfrac{B}{b}+\dfrac{C}{c}=0$.

But from (3), (4) and (5), by dividing (3) by a, (4) by b and (5) by c, we find that

$$\frac{A}{a}+\frac{B}{b}+\frac{C}{c}=\frac{a}{(a-b)(b-c)}+\frac{b}{(b-a)(b-c)}+\frac{c}{(c-a)(c-b)}=0.$$

224. The principle of Indeterminate Coefficients is often employed in finding the sum of a series.

Ex. 1.—Sum the series, $1^2+2^2+3^2+4^2+\ldots\ldots n^2$.

Assume

$$1^2+2^2+3^2+4^2+\ldots n^2=A_0+A_1n+A_2n^2+A_3n^3+A_4n^4+\ldots \quad (1)$$

Then, since the sum of the squares of the first n natural numbers is a function of n,

$$1^2+2^2+3^2+4^2+\ldots n^2+(n+1)^2$$
$$=A_0+A_1(n+1)+A_2(n+1)^2+A_3(n+1)^3+A_4(n+1)^4+\ldots \quad (2)$$

Subtracting (1) from (2),

$$(n+1)^2=A_1+A_2(2n+1)+A_3(3n^2+3n+1)+A_4(4n^3+6n^2+4n+1)+\ldots$$

or $$n^2+2n+1=A_1+A_2(2n+1)+A_3(3n^2+3n+1)+\ldots \quad (3)$$

It is not necessary to write down any coefficients beyond A_3, because A_4, A_5, $A_6\ldots$ are the coefficients of n^3, $n^4\ldots$ in (3), which do not exist on the left-hand side of this identity, and $\therefore$ must be $=0$.

Equating coefficients of the same powers of n,

$$1=A_1+A_2+A_3 \text{ (term without } n), \quad (1)$$

$$2=2A_2+3A_3 \text{ (coefficient of } n), \quad (2)$$

$$1=3A_3 \text{ (coefficient of } n^2). \quad (3)$$

Solving (1), (2) and (3) we obtain

$$A_3=\frac{1}{3},\quad A_2=\frac{1}{2},\quad A_1=\frac{1}{6}.$$

$$\therefore\ 1^2+2^2+3^2+4^2+n^2\ldots=A_0+\frac{1}{6}n+\frac{1}{2}n^2+\frac{1}{3}n^3.$$

Since this is true for all values of n, let $n=0$.

$$\therefore\ 0=A_0.$$

$$\therefore\ 1^2+2^2+3^2+4^2+\dots n^2=\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}$$

$$=\frac{n+3n^2+2n^3}{6}$$

$$=\frac{n(1+3n+2n^2)}{6}$$

$$=\frac{n(1+n)(1+2n)}{6}.$$

EXERCISE XXIII.

1. If a_1, a_2, a_3 be the roots of $x^3+px^2+qx+r=0$, express, in terms of p, q and r,

(1) $\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_3}$, (2) $a_1^3+a_2^3+a_3^3$, (3) $\dfrac{a_1}{a_2}+\dfrac{a_1}{a_3}+\dfrac{a_2}{a_3}+\dfrac{a_2}{a_1}+\dfrac{a_3}{a_1}+\dfrac{a_3}{a_2}$.

2. If a, b, c, d be the roots of $x^4-x^3+x+1=0$, find the values of

(1) $a^2b+a^2c+a^2d+b^2c+\dots.$ (2) $a^3+b^3+c^3+d^3$.

3. If a, b, c are the roots of $x^3+px^2+qx+r=0$, form the equations whose roots are (1) a^2, b^2, c^2; (2) ab, bc, ca.

4. Show that $x^3-5x^2+8x-4=0$ has two equal roots, and find all the roots.

5. If a, b, c be the roots of $x^3-2x^2+3x-4=0$, find the values of

(1) $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$, (2) $a^2+b^2+c^2$, (3) $\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}$.

6. Find the relation existing between p, q and r when the equation, $x^3+px^2+qx+r=0$, has two equal roots.

7. If two of the roots of $x^3+qx+r=0$ are equal, prove that $27r^2+4q^3=0$.

8. If a, b, c are the roots of $x^3+px^2+qx+r=0$, form the equations whose roots are (1) $b+c$, $c+a$, $a+b$; (2) b^2c^2, c^2a^2, a^2b^2.

9. The roots of $x^3+qx+r=0$ are denoted by a, b, c; form the equation whose roots are $ba+ac$, $cb+ba$, $ac+cb$.

10. Determine the relation between q and r necessary in order that the equation, $x^3+qx+r=0$, may be put in the form

$$x^4=(x^2+ax+b)^2,$$

and hence solve the equation, $8x^3-36x+27=0$.

11. Investigate the relation which exists between m and n when $mx^3-(2m^2+3n)x^2+(m^3+6mn)x-3m^2n$ is a perfect cube.

12. Determine the relations which exist among a, b, c, d, e, p, q when $ax^4+bx^3+cx^2+dx+e$ is divisible by x^2+px+q.

13. Investigate the condition for the expression,

$$4x^4-4px^3+4qx^2+2p(m+1)x+(m+1)^2,$$

being a perfect square.

14. Investigate the condition for the expression,

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F,$$

being divisible by a factor of the form $ax+by+c$.

15. Express $4(x^4+x^3+x^2+x+1)$ as the difference between two squares.

16. Investigate the relation between the coefficients that the equation, $ax^3+bx^2+cx+d=0$, may be put under one of the forms,

$$(1)\quad x^4=(x^2+px+q)^2,$$
$$(2)\quad q^2=(x^2+px+q)^2.$$

Solve in this way $2x^3-x^2-2x+1=0$.

17. If two of the roots of $ax^3+3bx^2+3cx+d=0$ are equal, prove $4(ac-b^2)(bd-c^2)=(ad-bc)^2$.

18. Show that $x^4+px^3+qx^2+rx+s$ can be resolved into two rational quadratic factors if s be a perfect square, negative, and equal to

$$\frac{r^2}{p^2-4q}.$$

19. If ax^3+bx^2+cx+d be a complete cube, show that $ac^3=db^3$ and $b^2=3ac$.

20. If $ax^4+bx^3+cx^2+dx+e$ be a complete fourth power, prove $bd=16ae$, $bc=6ad$ and $cd=6be$

21. If $px^3+3qx^2+3rx+s$ vanish when $x=a$ or b or c, express in terms of p, q, r, s,

(1) $a+b+c$, (2) $a^2+b^2+c^2$, (3) $a^3+b^3+c^3-3abc$.

22. From

$$\frac{x^3}{(x-a)(x-b)(x-c)(x-d)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}+\frac{D}{x-d},$$

prove
$$\frac{a^2}{(a-b)(a-c)(a-d)}+\frac{b^2}{(b-a)(b-c)(b-d)}$$
$$+\frac{c^2}{(c-a)(c-b)(c-d)}+\frac{d^2}{(d-a)(d-b)(d-c)}=0.$$

23. Determine the value of

$$\frac{a}{(a-b)(a-c)(a-d)}+\frac{b}{(b-a)(b-c)(b-d)}$$
$$+\frac{c}{(c-a)(c-b)(c-d)}+\frac{d}{(d-a)(d-b)(d-c)}.$$

24. Determine the value of

$$\frac{a^3}{(a-b)(a-c)(a-d)}+\text{anal}+\text{anal}+\text{anal}.$$

25. If $\dfrac{1}{(1-x)^2}=a_0+a_1x+a_2x^2+a_3x^3+\dots$, prove $a_0=1$, $a_1=2$, $a_2=3\dots$ and $a_n=n+1$.

26. If $\dfrac{1}{1-x} = a_0 + a_1x + a_2x^2 \dots,$ prove $a_0 = a_1 = a_2 \dots = 1.$

27. Determine a, b, c, d, e so that the n^{th} term in the expansion of $\dfrac{a + bx + cx^2 + dx^3 + ex^4}{(1-x)^5}$ may be n^4x^{n-1}.

28. Expand, in ascending powers of x,

$$\frac{1}{2-3x},\quad \frac{1+x}{2+3x},\quad \frac{1-x}{1-x+x^2}.$$

29. If $(1+x)^n = 1 + A_1x + A_2x^2 \dots$ and $(1+x)^{-n} = 1 + B_1x + B_2x^2 \dots,$ prove $A_3 + A_2B_1 + A_1B_2 + B_3 = 0.$

30. What values of x and y make the fraction,

$$\frac{2z^2 + (x-a)z + 2b(x-2c)}{3z^2 + (y-b)z + 3a(y-3c)},$$

independent of z?

31. Sum the series, $1 \,.\, 2 + 2 \,.\, 3 + 3 \,.\, 4 + \dots \, n(n+1).$

32. Sum the series, $1^3 + 2^3 + 3^3 + \dots \, n^3.$

33. Sum the series, $1 + 3 + 5 + \dots \, (2n-1).$

CHAPTER XVII.

PERMUTATIONS, COMBINATIONS AND DISTRIBUTIONS.

225. The word **Permutation** is used to designate a number of things arranged in a definite order.

Thus abc, acb, bca, bac, cba, cab are the permutations of the letters a, b, c taken all together. Sometimes we speak of the permutations of a number of things, of which only a part are taken at a time. Thus ab, ba, ac, ca, bc, cb are the permutations of the letters a, b, c, two being taken at a time.

226. The word **Combination** is used to designate a number of things taken as a whole, without regard to the order of their arrangement.

Thus abc, bcd, cda, dab are the combinations of four letters, a, b, c, d, three being taken at a time. The groups, abc, acb, etc., are different permutations, but the same combination.

227. The word **Distribution** is used to signify a mode of division of a number of things into parcels, or groups. In this connection we shall use the word **Parcel** to refer only to the things taken together, and the word **Group** when we wish to distinguish both things and order of arrangement.

228. When we speak of n things without further description of them, we shall assume that they are all different, *i.e.*, each is capable of being distinguished by the eye from every other. If

two or more are alike, the exact number of such like things will be specified; and in this case each individual thing will be counted in stating the total number. But when we speak of n things, each of which may be repeated any number of times, we mean n different kinds, with an unlimited number of each kind.

229. *If one action can be performed in m different ways, and if, after it has been done in any one way, another action can be performed in n different ways, then the two actions jointly can be performed in mn different ways.*

Let the first action be the selection of one of the capital letters, $A, B, C \ldots$, and the second the selection of one of the small letters, $a, b, c \ldots$, and let there be m of the first set and n of the second set. The letter A may first be chosen, and then any one of the n letters, $a, b, c \ldots$, making n choices in which A is taken first. Similarly there may be n choices in which B is taken first, and so on. Thus we have in all n choices repeated m times, *i.e.*, the two selections together may be made in mn different ways.

Cor.—This principle may easily be extended to three or more sets of operations. Thus if the first action can be performed in m ways, the second in n ways, and the third in p ways, the whole can be performed in mnp ways, and so on to any extent.

230. The preceding Art. contains the fundamental principle of the reasoning employed in this chapter. In applying it to any particular problem care must be taken to see that *all the results are different* and that *all* the *different cases are included.*

Ex. 1.—In how many ways can two persons be seated in a room containing 10 vacant seats?

One person can select any one of the 10 different seats, and then the other can take any one of the remaining 9, making in all $10 \times 9 = 90$ different ways.

Ex. 2.—In how many ways can the letters a, a, a, a, a, b, c be arranged in a line?

Place all the a's in a line with spaces between them. This can be done in only 1 way. Place the b at one end or in one of the four spaces. This can be done in 6 ways. The c can now be placed at either end or in one of the five spaces, making 7 ways. The total number of different arrangements is therefore $1 \times 6 \times 7 = 42$.

PERMUTATIONS.

231.—*To find the number of permutations of n things, r being taken at a time.*

Each of the n things may be followed in succession by each of the remaining $n-1$ things, giving $n(n-1)$ permutations of two things each. Each of these $n(n-1)$ permutations may be followed by each of the remaining $n-2$ things, giving $n(n-1)(n-2)$ permutations of three things each. Proceeding in the same way, and noting that the number of factors in each result is the same as the number of things taken together, we see that the number of permutations, r at a time, is

$$n(n-1)(n-2) \ldots . \text{ to } r \text{ factors,}$$

or $$n(n-1)(n-2) \ldots . (n-r+1), \text{ the result required.}$$

Cor.—The number of permutations of n things taken all at a time is

$$n(n-1)(n-2) \ldots . 3 . 2 . 1.$$

It is usual to denote this product by the symbol $\underline{|n}$, called "factorial n." In modern American works it is usually denoted by $n!$

The number of permutations of n things, r at a time, may conveniently be denoted by ${}_nP_r$.

232. *To find the number of permutations of n things, r at a time, when each thing may be repeated any number of times.*

Each of the n things may be followed in succession by one of its own kind and by one of each of the remaining kinds, giving n^2 permutations of two things each. Again, each of these n^2 permutations may be followed by any one of the n things, giving n^3 permutations of three things each. Proceeding in this way, and noting that the exponent of n is always the same as the number of things taken, we see that n^r is the number of permutations required.

Ex.—Seven travellers arrive at a town in which there are 10 hotels; in how many ways may they be accommodated with lodgings, provided all, or any number of them, may go to one hotel?

Each traveller may choose any one of the 10 hotels. Each choice of the first may be taken in connection with each choice of the second, making 10^2 arrangements for the first two. Each of these may be taken with each of the ten choices of the third, making 10^3 arrangements for the first three, etc. The 7 travellers may therefore be entertained at the 10 hotels in 10^7 or *ten million different ways.*

233. *To find the number of permutations of n things, all at a time, of which p are alike of one kind, q alike of another kind, r alike of another kind, and the rest all different.*

Let N denote the required number. Suppose all the possible permutations written out; then, if we place distinguishing marks upon each of the p like things, and permutate them in all possible ways, from each of the original numbers we can form $\lfloor p$ permutations without disturbing any of the other things, giving in all $N \times \lfloor p$ permutations. Similarly, by placing distinguishing marks upon each of the q like things we can form $\lfloor q$ permutations from each of those preceding. In like manner, by distinguishing the

r like things, we can form $\underline{|r}$ permutations from each of the latter. But the things are now all different, and consequently admit of $\underline{|n}$ permutations.

Therefore $$N \times \underline{|p} \times \underline{|q} \times \underline{|r} = \underline{|n},$$

or $$N = \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}}.$$

This process may evidently be continued to any extent.

234. *To find the number of ways in which n things can be arranged in a circle.*

Since only relative position is considered we shall get all possible arrangements by placing one thing in position and permutating the other $n-1$ things as in a straight line, giving in all $\underline{|n-1}$ permutations.

These permutations may be arranged in pairs, the order of the things when proceeding from right to left in the one being the same as in proceeding from left to right in the other. If these be considered alike, the preceding result must be divided by 2, giving $\frac{1}{2}\underline{|n-1}$ as the result required.

It may appear to the student, at first thought, that the result should be the same as if the things were arranged in a straight line, viz., $\underline{|n}$. If so, place each of the following permutations of four letters, *abcd*, *bcda*, *cdab*, *dabc*, around a circle, when it will be found that though they are different permutations in a line, they form the same arrangement in a circle. Similarly all the permutations of n letters may be arranged in groups of n permutations each, which give the same arrangement in a circle. The whole number of circular arrangements is therefore $\underline{|n} \div n$ or $\underline{|n-1}$.

EXERCISE XXIV.

1. How many different numbers, each consisting of three figures, can be formed with the nine digits?

2. In how many ways can a consonant and a vowel be chosen from the word *p e r m u t a t i o n*?

3. A basket contains 20 oranges and 15 pears. In how many ways can one of each be chosen? In how many ways can two persons each choose one?

4. How many dictionaries must be published to translate any one of 5 languages into any one of the other 4? How many new ones will be required to include 3 more languages?

5. How many of the permutations of the 5 vowels, taken all together, begin with *ae*? In how many will *ae* occur together?

6. If all the numbers which can be formed with the 9 digits, taken all together, be written out, how many of them will begin with 312 and end with 89?

7. In how many ways can 10 persons be seated (1) upon 10, (2) upon 11, (3) upon 12 chairs placed in a row?

8. How many different signals can be made with 5 flags of different colors, 3 flags being used in each signal and all placed in a horizontal line? How many by placing any number of them in a horizontal line?

9. How many different signals can be made with 20 flags of 5 different colors, 4 of each color, any number less than 5 being used, but always placed in a straight line, either vertical or horizontal?

10. The number of permutations of m things, three at a time, is $\frac{2}{3}$ of the number of permutations of $m+1$ things, three at a time; find m.

11. The number of permutations of m things, three at a time, is $\frac{11}{17}$ of the number of permutations of $4m$ things, two at a time; find m.

12. How many permutations can be made from the letters of the word *facetiously*, taken all together, (1) by permutating the vowels only; (2) by keeping all the vowels together; (3) by keeping all the vowels in the given order?

13. How many permutations can be made from the letters of the following words, the letters of each word being taken all together, *Mississippi*, *proportion*, *indivisibility*?

14. From a party of 12 ladies and 10 gentlemen one lady and one gentleman are to be chosen; in how many ways may this be done so that no one of three specified ladies may be chosen with either of two specified gentlemen?

15. Five ladies and 5 gentlemen drive out in 5 separate carriages, one lady and one gentleman in each; in how many ways may the party be arranged, including the order of the carriages?

16. In how many ways may 5 speakers be called upon, (1) providing B may not speak before A; (2) if A must be the third speaker and B may not speak before him?

17. By how many different ways may a student go from his home to school who lives 4 blocks to the north and 3 to the east from the school-house?

18. In how many different ways can 5 apples and 5 oranges be distributed among 10 boys, giving each boy one, supposing the apples to be alike, but the oranges to be different?

19. In how many ways can the letters of *ubiquitous* be arranged so that q may always be followed, (1) by u; (2) by only one u; (3) by just two u's?

20. On a shelf are 20 books, of which 8 form a set; in how many ways can they be arranged, (1) keeping the set in order

and unbroken; (2) keeping them in order, but allowing other books between; (3) keeping the set together, but not in order?

21. In how many ways can 5 books be arranged on a shelf if any book may be placed either end up and either side to the front?

22. How many signals can be formed with 20 flags of different colors, not more than 4 flags being used to form a signal, and being placed in a line vertically, horizontally or diagonally?

23. Tom, Dick and Harry scramble for an apple, an orange and a pear; in how many different ways may they pick them up? In how many ways if the three things were all alike, or if two were alike?

24. In how many ways can 10 persons form a ring so that a certain couple may always be beside each other?

25. In how many different ways may 8 persons be seated at a round table, the seats being distinguished? In how many ways may 8 children form a ring? In how many ways may 8 different beads be made into a bracelet?

26. In how many ways can 7 persons sit at a round table so that the host may have the guest of highest rank on his right, and the next in rank on his left?

27. In how many ways can a company of 12 sit at a round table so that the host and hostess may sit opposite each other?

28. In how many ways can a party of 10 form a ring so that a specified couple shall never be beside each other?

29. On a shelf are placed 5 Latin, 3 Greek, 4 French and 6 English books. In how many ways may they be arranged, (1) keeping those of each kind together; (2) keeping each set in order from left to right or from right to left, but allowing other books to be placed between?

30. In how many ways can the letters of the word *syzygy* be arranged, (1) with the three *y*'s together; (2) with each *y* separate; (3) just two *y*'s together?

31. In how many different ways can the letters in the word *indivisibility* be arranged, (1) with all the i's together; (2) with just five i's together?

32. In how many ways can m ladies and m gentlemen form a ring so that no two ladies shall be together?

33. Twenty male and 6 female candidates apply to a school board who have to fill 10 different situations, 4 of which must be held by men and 3 others by women; in how many different ways may the appointments be made?

COMBINATIONS.

235. *To find the number of combinations of n things, r being taken at a time.*

Let N denote the required number. Now from each combination $\underline{|r}$ permutations may be made, making in all $N \times \underline{|r}$ permutations; but this will evidently be the total number of permutations of n things, r at a time.

Therefore $$N \times \underline{|r} = n(n-1)(n-2)\ldots\ldots(n-r+1),$$

or $$N = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\underline{|r}},$$

the number of combinations required.

If we multiply both numerator and denominator of this fraction by $(n-r)(n-r-1)\ldots\ldots 3 \cdot 2 \cdot 1$, *i.e.*, by $\underline{|n-r}$, we may write the result in the neat form,

$$N = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

The symbol ${}_nC_r$ is frequently used to denote the number of combinations of n things, r at a time.

236. *The number of combinations of n things, r at a time, is equal to the number of combinations of n things, $n-r$ at a time.*

This is at once evident from the fact that when r things have been selected to form a combination, $n-r$ things are left to form a corresponding combination. The proof also follows easily from the formulæ thus:

$$_nC_{n-r} = \frac{n(n-1)\,.\,.\,n-(n-r)+1}{\lfloor n-r} = \frac{n(n-1)\,.\,.\,r+1}{\lfloor n-r} \times \frac{\lfloor r}{\lfloor r} = \frac{\lfloor n}{\lfloor n-r \lfloor r}.$$

which is the result obtained for $_nC_r$.

237. *To find the total number of combinations which can be made from n things, any number being taken at a time.*

In proceeding to form a combination each thing in succession may be disposed of in two ways, *i.e.*, it may be either taken or left; and since either mode of dealing with any one thing may be followed by either mode of dealing with each of the others in succession, the total number of ways is

$$2 \times 2 \times 2 \dots \text{ to } n \text{ factors.}$$

But this includes the case in which *all* are rejected. The total number of combinations is therefore $2^n - 1$.

From this article we get an indirect proof of the following equation:

$$_nC_1 + {}_nC_2 + {}_nC_3 + \dots {}_nC_n = 2^n - 1,$$

which may easily be verified for any particular value of n.

238. *To find the number of combinations which can be formed from a collection of things of which p are alike of one kind, q alike of a second kind, r alike of a third kind, and so on, any number being taken at a time.*

The p things may be disposed of in $p+1$ ways, since we may take $0, 1, 2, 3 \dots p$ of them. Similarly the q things may be dis

posed of in $q+1$ ways, and so on. The total number of ways of disposing of all the things is therefore

$$(p+1)(q+1)(r+1)\ldots\ldots$$

But this includes the case in which all the things are rejected. Therefore the total number of combinations is

$$(p+1)(q+1)(r+1)\ldots\ldots-1.$$

239. *To find the number of combinations of n things, r at a time, when each thing may be repeated any number of times.*

Denote the n things by the natural numbers, $1, 2, 3 \ldots\ldots n$. Take any combination of r of these numbers (with repetitions), arrange them in numerical order, and to the numbers of this series add the numbers $0, 1, 2, 3 \ldots\ldots r-1$ respectively. The resulting numbers must be found in the series, $1, 2, 3 \ldots\ldots n+r-1$. Conversely, from this latter series select any combination of r numbers, arrange them in ascending order of magnitude, and from them subtract $0, 1, 2, 3 \ldots\ldots r-1$ respectively. The resulting numbers must form a combination (with repetitions) of r numbers no one of which is greater than n.

Hence for every combination of r out of n things with repetitions there is a corresponding combination of r out of $n+r-1$ things without repetitions, and conversely. The number of the latter combination is

$$\frac{(n+r-1)(n+r-2)\ldots\ldots n}{\underline{|r}} \quad \text{or} \quad \frac{\underline{|n+r-1}}{\underline{|r}\;\underline{|n-1}},$$

which is therefore the number required.

If we denote the different things by the letters a, b, c, etc., and the number of each found in any one combination by an exponent, then if all the combinations be written out we shall have all the terms of r dimensions that can be formed from the n letters. Hence this proposition is often quoted as that of finding *how many homogeneous products of r dimensions can be formed from n symbols.*

240. The proposition of the preceding Art. may also be proved in the following manner, which will be an instructive exercise for the student:

Suppose all the combinations written out. Denote their number by

$$C_r.$$

In each combination there are r letters, therefore each letter must be repeated

$$\frac{r}{n}C_r$$

times in the whole number of combinations.

Again, if any one letter, a for example, be removed a single time from every combination in which it is found, the resulting combinations will be those which can be formed from the n letters, $r-1$ at a time; and the number of times in which a will be repeated in them is

$$\frac{r-1}{n}C_{r-1},$$

and the a has been removed *once* from each combination; therefore the total number of times in which a enters into the original combinations is

$$\frac{r-1}{n}C_{r-1} + C_{r-1},$$

and this must equal the number formerly found.

Equating the two expressions we get

$$\frac{r}{n}C_r = \frac{r-1}{n}C_{r-1} + C_{r-1}$$

$$= \frac{n+r-1}{n}C_{r-1}.$$

Therefore $$C_r = \frac{n+r-1}{r}C_{r-1}.$$

Similarly
$$C_{r-1}=\frac{n+r-2}{r-1}C_{r-1},$$
$$C_{r-2}=\frac{n+r-3}{\underline{|r-2}}C_{r-3}$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$C_2=\frac{n+1}{2}C_1$$
$$=\frac{n+1}{2}\cdot\frac{n}{1}.$$

Multiplying these equations and cancelling like terms on the opposite side of the resulting equation we get

$$C_r=\frac{(n+r-1)(n+r-2)\ldots\ldots(n+1)n}{r(r-1)\ldots\ldots2\,.\,1}$$
$$=\frac{\underline{|n+r-1}}{\underline{|r}\;\underline{|n-1}}.$$

241. *To find the number of combinations of n things of which p are alike, r being taken at a time.*

I. Suppose r not greater than p.

From the p like things one combination of r like things may be formed; with $r-1$ like things take each of the remaining $n-p$ unlike things in succession, giving $n-p$ combinations; with $r-2$ like things take each combination of two of the $n-p$ unlike things, giving

$$\frac{(n-p)(n-p-1)}{\underline{|2}}$$

combinations. Proceeding in the same way we find the whole number of combinations to be

$$1+(n-p)+\frac{(n-p)(n-p-1)}{\underline{|2}}+\frac{(n-p)(n-p-1)(n-p-2)}{\underline{|3}}+\text{etc.},$$

the last term in the series being that in which all the unlike things are employed.

II. Suppose r greater than p.

Take the p like things and $r-p$ of the unlike things. This can be done in

$$\frac{\lfloor n-p}{\lfloor r-p \lfloor n-r}$$

ways, and continue the series as before.

Ex.—Find the number of combinations which may be formed from the letters *aaaabcdef*, (1) taking three at a time, (2) taking five at a time.

1. Form one combination from the a's alone, viz., *aaa*. Next take two a's and one other letter, giving 5 combinations. Next take one a and two other letters, giving $\frac{5.4}{1.2}$ or 10 combinations. Lastly, without an a we can form $\frac{5.4.3}{1.2.3}$ or 10 combinations. The total number is therefore $1+5+10+10=26$ combinations.

2. Forming the combinations of b, c, d, e, f in succession, 1, 2, 3, 4 and 5 at a time, and with each combination placing the proper number of a's, we get all possible combinations, 5 letters at a time. The numbers are $5+10+10+5+1=31$, the number of combinations required.

242. Problems occasionally present themselves in which it is required to find the number of combinations (or permutations) of n things, r at a time, when there are several sets of like things to be chosen from. No general formulæ can be given for such cases, but the following example will indicate the method to be pursued in any particular problem:

Ex.—Find the total number of combinations which can be formed from the letters in the word *p r o p o r t i o n* taken 6 at a time.

This problem must be divided into six parts, as follows:

(1) With all the letters different we obtain . . . . . 1 comb'n.
(2) With one double letter we get . . . . . . $3 \times 5 = 15$ comb'ns.
(3) With two double letters we get. . . . . . $3 \times 6 = 18$ comb'ns.
(4) With three double letters we get . . . . 1 comb'n.
(5) With three o's and the rest different we get . . . 10 comb'ns.
(6) With three o's and one double letter we get $2 \times 4 = 8$ comb'ns

The sum of these numbers is 53, the number of combinations required.

The number of permutations, 6 at a time, may easily be found from the preceding; for the first combination will produce $\underline{|6}$ permutations; each combination in (2) will furnish $\frac{\underline{|6}}{\underline{|2}}$ permutations, since there are two letters alike in each, and so on. The total number is 11,130.

243. The theorem given in the next Art. usually presents a considerable difficulty to beginners; we therefore give a numerical illustration to prepare the way for a general proof.

Consider the number of combinations of 10 things taken 1, 2, 3, 4, etc., at a time. We have

$$C_1 = \frac{10}{1}, \quad C_2 = \frac{10.9}{1.2}, \quad C_3 = \frac{10.9.8}{1.2.3}, \quad C_4 = \frac{10.9.8.7}{1.2.3.4},$$

$$C_5 = \frac{10.9.8.7.6}{1.2.3.4.5}, \quad C_6 = \frac{10.9.8.7.6.5}{1.2.3.4.5.6}, \text{ etc.}$$

Now, observe that each combination is formed from the previous one by placing one more factor in both numerator and denominator; and therefore each result is greater than the preceding so long as the factor in the numerator is greater than the corresponding one in the denominator. In this example C_5 is the

greatest, being greater than any one of the preceding; and of those which follow,

$$C_6 = \frac{5}{6} C_5, \quad C_7 = \frac{4}{7} C_6, \text{ etc.,}$$

which shows that from C_5 each number is less than the preceding.

In like manner write out the number of combinations in succession of 9 things, when it will be observed that the numbers increase up to C_4; that C_4 and C_5 are equal; and that these are greater than any others.

From these two examples it may be perceived that when n is *even*, the greatest number of combinations can be formed by taking $\frac{n}{2}$ at a time; but when n is *odd*, two results, viz., those formed by taking $\frac{n-1}{2}$ and $\frac{n+1}{2}$ at a time, are equal, and that these are greater than any other.

244. *To find the value of r for which the number of combinations of n things, r at a time, is greatest.*

With the usual notation we have

$${}_nC_r = {}_nC_{r-1} \times \frac{n-r+1}{r}.$$

Therefore $\qquad {}_nC_r >, =, \text{ or } <, {}_nC_{r-1},$

according as $\qquad \dfrac{n-r+1}{r} >, =, \text{ or } <, 1;$

i.e., as $\qquad n-r+1 >, =, \text{ or } <, r;$

or as $\qquad n+1 >, =, \text{ or } <, 2r;$

or as $\qquad r <, =, \text{ or } >, \dfrac{n+1}{2}.$

Now, from the nature of the problem, r must be a positive integer; therefore

(1) If n be even, ${}_nC_r$ is greatest when $r = \frac{n}{2}$.

(2) If n be odd, ${}_nC_r = {}_nC_{r-1}$ when $r = \frac{n+1}{2}$, and in these two cases the number of combinations is greater than in any other, *i.e.*, ${}_nC_r$ is greatest when $r =$ either $\frac{n-1}{2}$ or $\frac{n+1}{2}$.

245. *To express the number of combinations of n things, r at a time, in terms of the number of combinations for smaller values of n and r.*

The total number of combinations is evidently made up of those which can be formed without including a particular thing, together with those each of which does include it. The number of the former is ${}_{n-1}C_r$; the number of the latter is ${}_{n-1}C_{r-1}$, for any specified thing will appear as many times as different combinations of $r-1$ things can be formed from the remaining $n-1$ things to place with it. Each of these combinations can be separated into two parts in the same way, and so on to any extent. The process may be expressed in symbols as follows:

$$\begin{aligned} {}_nC_r &= {}_{n-1}C_r + {}_{n-1}C_{r-1} \\ &= {}_{n-2}C_r + 2{}_{n-2}C_{r-1} + {}_{n-2}C_{r-2} \\ &= {}_{n-3}C_r + 3{}_{n-3}C_{r-1} + 3{}_{n-3}C_{r-2} + {}_{n-3}C_{r-3}. \end{aligned}$$

EXERCISE XXV.

1. How many different parties of 4 can be chosen from 10 persons? In how many of these would a particular person be found?

2. How many different parties of 6 can be chosen from 20 persons? In how many of these would two particular persons be found? In how many would the first be present and the second absent?

3. In Arithmetical Progression there are five quantities concerned, and when any three are given the other two can be found. How many different formulæ can be given on the subject?

4. From 20 ladies and 15 gentlemen how many different parties of 5 ladies and 5 gentlemen can be formed? If the parties be considered different when different ladies and gentlemen are partners, how many parties can be formed?

5. From a company of 9 men and their wives a party of 4 men and 4 women are to be chosen. In how many ways can this be done so that a man and his wife shall not be in the same party?

6. In a basket are 5 apples at two for a cent, and 3 pears at a cent each. A boy having 2 cents in his pocket wants some fruit. How many choices can he make?

7. From 10 different books in how many ways may a choice of one or more books be made? If all possible choices be made in succession, how many times will any one book be chosen?

8. How many different sums of money can be made up by taking one or more of the following sums: 10 dollars, 1 dollar, 50 cents, 25 cents, 10 cents, 5 cents, 1 cent? What is the total value of all the sums thus formed? How many sums could be formed by using a 20-cent piece in addition?

9. In a basket are 10 oranges, 8 pears and 7 apples. How many different choices of a quantity of fruit can be made, the specimens of each kind of fruit being alike?

10. Apply Art. 238 to find the number of divisors of the numbers 540, 720.

11. Find the number of combinations of the letters of the word *indivisibility*, 4 letters being taken at a time. Find also the number of permutations 4 at a time, and the number of permutations all at a time, in which two *i*'s do not come together.

12. Find the number of combinations, (1) 5 at a time, (2) 6 at a time, of the letters of the words *ever esteemed friend*. Find also the number of permutations in each case.

13. In a basket are 25 oranges worth 3 cents each. How much money should a boy spend so as to have the greatest number of choices?

14. From 15 ladies and 8 gentlemen a committee of 7 is to be chosen. How many of each should be taken to permit of the greatest number of choices?

15. There are 17 consonants and 5 vowels. How many from each must be taken to form the greatest number of combinations containing a fixed number of both vowels and consonants? and how many such combinations can be formed?

16. In the preceding example how many combinations can be formed, each containing a fixed number of letters? What is the total number of combinations which can be formed, and how many of these will contain at least one vowel and two consonants?

17. Of $2n$ things n are alike and the rest are different. How many different combinations of n things each can be formed?

18. The number of combinations of n things, 5 together, is $3\frac{3}{5}$ times the number, 3 together. Find n.

19. The number of combinations of $2n$ things, 3 together, is $1\frac{2}{3}$ times the number of permutations of n things, 3 together. Find n.

20. The number of combinations of n things, r at a time, is the same as the number $2r$ at a time, and $2\frac{1}{4}$ times the number of combinations, $r-1$ at a time. Find n and r.

21. In how many ways may m boys and n girls form a ring so that no two girls shall be beside each other? $(m > n.)$

22. Three aldermen are to be elected from 5 candidates. In how many ways can 4 electors cast their votes, each elector having the privilege of voting for 1, 2 or 3 candidates?

23. The number of ways of selecting x things out of $2x+2$ is to the number of ways of selecting x things out of $2x-2$ as 99 to 7. Find x.

24. From n things, p of which are alike of one kind and q alike of another kind, how many choices of one or more things may be made?

25. How many different throws may be made with 2 dice? with 3 dice? with n dice?

26. In how many ways can a boy select a dozen marbles in a shop where there are 5 kinds for sale?

27. If $(a+b+c+d)^5$ be expanded, how many terms will there be in the result?

28. Out of 21 consonants and 5 vowels how many words, each containing 5 consonants and 3 vowels, can be formed?

29. Find the total number of permutations that can be formed from m things of one kind and n things of another kind, taking r of the former and s of the latter to form each permutation.

30. Find the number of signals which can be made with 4 lights of different colors when displayed any number at a time, arranged perpendicularly, horizontally or diagonally.

31. How many apples must be put in a basket with 9 oranges and 14 pears so that a person wishing to purchase some fruit may have 2,999 choices?

32. If n straight lines of indefinite length be drawn upon a plane, no two being parallel and no three passing through the same point, (1) how many intersections will there be? (2) how many triangles?

33. If n points in a plane be joined in all possible ways by indefinite straight lines, no two of which are coincident or parallel, and no three passing through any point except the original n points, (1) how many lines will there be? (2) how many triangles having their angular points on the original points? (3) how many triangles in all? (4) how many intersections, exclusive of the intersections at the n points?

34. There are n points in a plane, p of which are in a straight line. (1) How many straight lines can be formed by joining the points? (2) how many triangles will have their angular points on the original points?

35. If there be n straight lines in a plane, no three of which

meet at a point, find the number of groups of n of their points of intersection in each of which no three points lie in one of the straight lines.

36. In a plane are m straight lines which all pass through a given point, n others which pass through another point, and p others which pass through a third point. Supposing no other three to intersect in one point, and that no two are parallel, find the number of triangles formed by the intersection of the straight lines.

37. In an ordinary checker-board how many squares can be formed by grouping together any number of the original squares? How many could be made if there were n squares on each side of the original board?

38. If a cubic foot were divided into cubic inches, how many cubes could be formed by grouping together any number of the cubic inches without disarranging any of the small cubes? How many could be formed if the edge of the original block were n inches?

DISTRIBUTIONS.

246. In the problems which we have discussed in the previous sections of this chapter we found two chief elements for consideration, viz., the order of arrangement of things in a group and the particular things to be taken to form a group, in both cases the number of things in a group being given. But there are a number of other elements which, when taken into consideration, very much change the character of a problem, some of which are treated in the remainder of the chapter.

The general problem is the separation of a number of things into a series of classes. A very great variety exists in the particular problems which may be proposed, some of which are of considerable difficulty. The principal elements on which the distribution depends are five in number, as follows:

1. The things to be distributed may be alike or different.

2. The classes when formed may be distinguished from each other by some consideration independent of the elements they contain, or they may not be so distinguished.

3. The order of the things in a class may or may not be considered, *i.e.*, they may form a *group* or a *parcel*.

4. Blank groups or parcels may or may not be allowed.

5. Some of the things may or may not remain undistributed.

247. *To find the number of ways in which n things can be divided into two parcels containing r and $n-r$ things respectively.*

This is essentially the same as finding the number of combinations of n things, r at a time; for whenever a combination of r things is formed, another combination of $n-r$ things is left, and the original number is divided into two parcels. The result is therefore

$$\frac{\lfloor n}{\lfloor r \; \lfloor n-r}.$$

If $r=n-r$, and if there is no way of distinguishing the parcels except by the elements they contain, this result must be divided by 2.

Ex.—Four different books may be equally divided between 2 boys in 6 different ways; but they can be wrapped in parcels of 2 each in only 3 different ways.

248. *To find the number of ways in which n things may be divided into three parcels containing r, s and $n-r-s$ things respectively.*

From the n things r things may be selected in

$$\frac{\lfloor n}{\lfloor r \; \lfloor n-r}$$

different ways; and when any one selection has been made the remaining $n-r$ things may be divided in

$$\frac{\lfloor n-r}{\lfloor s \, \lfloor n-r-s}$$

ways. The two operations may therefore be performed in

$$\frac{\lfloor n}{\lfloor r \, \lfloor n-r} \times \frac{\lfloor n-r}{\lfloor s \, \lfloor n-r-s} = \frac{\lfloor n}{\lfloor r \, \lfloor s \, \lfloor n-r-s}$$

different ways. This process may evidently be continued to any extent.

If $r=s=n-r-s$, and if there is no method of distinguishing the parcels except by the elements they contain, this result must be divided by $\lfloor 3$; and if two of the three parcels contain an equal number of things, the result must be divided by 2, as in the former Art.

Ex.—Six persons may be placed in 3 different carriages, 2 in each carriage, in 90 different ways; but 6 different letters can be divided into 3 sets of 2 each in only 15 different ways.

249. *To find the number of ways in which n different things may be divided into r distinguished parcels (blanks allowed).*

The first thing may be placed in any one of the r different parcels; and when this has been done the second may also be placed in any one of the r parcels, giving r^2 ways of disposing of two things. Each of these ways may be followed by r ways of placing the third thing, making r^3 ways of disposing of three things. Proceeding in this way we see that n things can be divided into distinguished parcels in r^n ways.

Ex.—This proposition gives the number of ways in which n different prizes may be awarded to r students.

250. *To find the number of ways in which n different things can be arranged in r different groups (blanks allowed).*

The first thing may be placed in any one of r different groups; the second may be placed on either side of the first one, or in any one of the remaining $r-1$ groups, making $r+1$ different positions. Similarly the third may be placed in any one of $r+2$ different positions, and so on. Therefore all together there are

$$r(r+1)(r+2)\ldots.(r+n-1)$$

different arrangements.

Ex.—If a lady has 3 different rings, each of which may be worn on any finger of either hand, she can place them on her fingers in $8 \cdot 9 \cdot 10 = 720$ different ways.

251. *To find the number of ways in which n like things may be divided into r distinguished parcels (blanks allowed).*

Denote the parcels by A, B, C, etc., and the number of things placed in each by a, b, c, etc.; then we shall get such results as $A^aB^bC^c\ldots.$ for the different ways of division. Now, the only restriction is that $a+b+c+\ldots.=n$, since all the things must be distributed. This is evidently the same as forming all the homogeneous products of n dimensions from the r symbols, A, B, C, etc. The result is therefore

$$\frac{(r+n-1)!}{n!\,(r-1)!}. \qquad \text{Art. 239}$$

Ex.—If n marbles be thrown on the ground to be scrambled for by r boys, this proposition gives the number of ways in which they may be picked up.

252. *To find the number of ways in which n different things can be arranged in r different groups (no blanks).*

Arrange the n things in a line; we have then to insert $r-1$

points of division among the $n-1$ spaces between the n things. This can be done in

$$\frac{|\underline{n-1}}{|\underline{r-1}\;|\underline{n-r}}$$

ways. The things can be arranged in a line in $|\underline{n}$ different ways; therefore the required number is

$$\frac{|\underline{n}\;|\underline{n-1}}{|\underline{r-1}\;|\underline{n-r}}.$$

Ex.—The number of ways in which n cars can be attached to r different engines, one car at least being attached to each engine, is

$$\frac{|\underline{n}\;|\underline{n-1}}{|\underline{r-1}\;|\underline{n-r}}.$$

253. *To find the number of ways in which n like things can be divided into r distinguished parcels (no blanks).*

Place the n things in a row. Since all are alike this can be done in only one way. Insert $r-1$ points of division among the $n-1$ intervals, and place the things between the successive points in the parcels in order. The result is evidently the number of combinations of $n-1$ things, $r-1$ at a time; that is,

$$\frac{|\underline{n-1}}{|\underline{r-1}\;|\underline{n-r}}.$$

Ex.—This proposition gives the number of ways in which n marbles, all alike, may be distributed among r boys so that each boy gets at least one.

The preceding propositions do not, by any means, exhaust the number of problems which may be proposed in this part of the subject; but they contain a sufficient variety for the ordinary student. Those who desire to pursue the subject further, and to investigate the more intricate problems, are referred to the very

excellent work entitled, "Choice and Chance," by William Allan Whitworth, which has been consulted to some extent in the preparation of this chapter.

EXERCISE XXVI.

1. In how many ways can 6 different books be equally divided between 2 boys? In how many ways can they be divided into 2 equal parcels?

2. In how many ways can 12 persons be divided into 3 equal companies? In how many ways can they be placed in 3 different carriages, 4 in each carriage?

3. In how many ways may the 26 letters of the alphabet be divided into 5 parts, 4 of the parts containing 6 letters each?

4. In how many ways can a selection of $(n-1)r$ things be made from $n-1$ sets, containing $2r, 3r \ldots nr$ things respectively, taking r things from each set?

5. How many signals can be made with 7 differently colored flags on 5 masts, all the flags being used, but not necessarily all the masts?

6. In how many ways can 20 different cars be attached to 4 different engines, any number being attached to an engine?

7. In how many ways may 10 apples, all alike, be divided between 5 boys, any number being given to a boy?

8. At a matriculation examination 100 students compete for 5 scholarships. In how many ways may they be awarded, (1) if the award is made for general proficiency? (2) if each scholarship is awarded for one department only?

9. How many signals can be made with 7 different flags on 5 masts, if all the flags must be used and every mast must have a flag?

10. In how many ways can the letters of the alphabet be made into 6 words, each letter being used once, and only once?

11. In how many ways can an examiner assign 100 marks to 10 questions, some value being given to each question?

12. Twenty shots are to be fired. In how many ways may the work be distributed among 4 guns, (1) without leaving any gun unemployed; (2) without any restriction?

13. If n marbles, all alike, are thrown on the ground to be scrambled for by r boys, in how many ways may they be picked up, (1) if all the marbles are found; (2) if some of them are lost?

14. In a ladies' school are 15 pupils, who walk out in 5 rows with 3 in each row. In how many different ways can they be arranged so that no two shall be twice in the same row?

15. There are $3n+1$ things, of which n are alike and the rest all different. In how many ways may n things be selected from them?

16. In how many ways can 3 numbers in arithmetical progression be selected from an arithmetical series of n terms?

17. How many different selections of $2n$ things may be made from a collection of n like things of one kind, n like things of a second kind, and $2n$ like things of a third kind?

18. A, B and C have respectively the letters of *p r o p o r t i o n*, *s q u a r e r o o t* and *l o g a r i t h m s*. In how many ways may they exchange so that each will still have 10 letters, but only one of the persons will have two or more letters alike?

19. A square is divided into 16 equal squares by vertical and horizontal lines. In how many different ways may 4 of these be painted white, 4 black, 4 red and 4 blue, without repeating a color in the same vertical or the same horizontal line?

20. Show that

$$1+\frac{m}{1}+\frac{m(m+1)}{1\,.\,2}+\frac{m(m+1)(m+2)}{1\,.\,2\,.\,3}+\dots \text{ to } n+1 \text{ terms}$$

$$=1+\frac{n}{1}+\frac{n(n+1)}{1\,.\,2}+\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}+\dots \text{ to } m+1 \text{ terms.}$$

21. A debating society has to select 1 out of 5 subjects proposed. If 30 members vote, each for 1 subject, in how many ways can the votes fall?

22. If $pq+r$ things are to be divided as equally as possible among p persons, in how many ways can it be done? $(r<p.)$

23. In how many ways can p positive and n negative signs be placed in a row so that two negative signs shall not come together? $(p =$ or $> n.)$

24. In how many ways may the letters of *permutations*, *combinations*, *distribution*, be divided among 3 persons, giving 6 double letters to one person, and 12 letters, no two being alike, to another?

25. In how many ways may $2n$ like things of each of 3 different kinds be divided between 2 persons so that each person may have $3n$ things?

26. In how many ways may the letters of *Higher Algebra* be divided among 3 persons, any number of letters being given to one person?

27. The game of bagatelle is played with 8 balls all alike and 1 different. The object is to get as many as possible of the balls into 9 different holes, each of which is capable of receiving but 1 ball. How many different arrangements of the balls are possible?

28. If bagatelle is played with n balls alike and one different, and there are $n+1$ holes, each capable of receiving one ball, the whole number of ways in which the balls can be disposed of is $(n+3)2^n-1$.

CHAPTER XVIII.

MATHEMATICAL INDUCTION.

254. Suppose we were required to find the sum of the series

$$1 + 3 + 5 + \ldots .$$

to n terms. This might be done by the following line of reasoning:

1. By observation and trial we find that

$$1 + 3 = 2^2, \quad 1 + 3 + 5 = 3^2, \quad 1 + 3 + 5 + 7 = 4^2.$$

Here it is seen that the sum of *two* terms $= 2^2$, the sum of *three* terms $= 3^2$, and the sum of *four* terms $= 4^2$. These facts would indicate or suggest that the sum of n terms $= n^2$.

2. To prove that this is so let us *assume* that

$$1 + 3 + 5 + 7 + \ldots . (2n - 1) = n^2,$$

and then proceed to examine what will follow from this *assumption*. Adding $(2n + 1)$ to both sides we obtain

$$1 + 3 + 5 + 7 + \ldots . (2n - 1) + (2n + 1) = n^2 + 2n + 1 ;$$

that is, the sum of $(n + 1)$ terms $= (n + 1)^2$.

3. It is now proved that if the law holds for the sum of r terms, it holds for the sum of $(n + 1)$ terms. By trial it was found that it does hold for *four* terms, therefore it *must* hold for *five* terms; and holding for *five* terms it must hold for *six* terms, and so on. Hence we conclude that the sum of the first n odd natural numbers always equals n^2.

255. This method of proof, or line of reasoning, is called **Mathematical Induction,** from its resemblance to that mode of reasoning called "Induction" employed in the discovery of new truths. It differs from Induction, as applied to the discovery of new physical truths, in that it leaves no room for doubt, and that the conclusion we reach is no wider than the premises. To a certain class of problems, many of them connected with series, Mathematical Induction is the only line of investigation and proof available, and its conclusions are as rigidly true as those reached by any other method of investigation. Nevertheless there is generally a feeling of dissatisfaction in the mind of the student when first called upon to reach general results this way. The various steps taken are fairly illustrated in the example we have just used. There is:

(1) The *probable* discovery of the *law* relating to the quantities under consideration.

(2) For the purpose of further investigation the *assumption* that the law holds good for a *general* value, n, of one or more of the quantities involved.

(3) The *proof* that if this law holds good for the value n, it must hold good for the next value, $(n+1)$.

(4) By trial it is seen to hold good for a particular numerical value, and therefore by (3) it must hold good for the numerical value greater by one. Following out this line of reasoning we see that the result, or law, is generally true.

In many problems the law is given, and its proof alone required; but in its general application the law has to be discovered as well as proved to be true.

256. The following are further illustrations of the application of Mathematical Induction:

Ex. 1.—Prove that when n is positive and integral, $x^n - a^n$ is divisible by $(x - a)$.

$$\frac{x^n - a^n}{x - a} = x^{n-1} + \frac{a(x^{n-1} - a^{n-1})}{x - a}.$$

Therefore $x^n - a^n$ is divisible by $(x-a)$ if $x^{n-1} - a^{n-1}$ is divisible by $(x-a)$.

By trial we find that when $n=3$, $x^n - a^n$ is divisible by $(x-a)$; $\therefore$ $x^4 - a^4$ is also divisible by $(x-a)$, and $\therefore$ $x^5 - a^5$, $x^6 - a^6 \dots\dots$ $x^n - a^n$ is always divisible by $(x-a)$ when n is positive and integral.

Ex. 2.—If $a_1, a_2, a_3 \dots\dots a_n$ be in H. P., prove

$$a_1a_2 + a_2a_3 + \dots\dots a_{n-1}a_n = (n-1)a_1a_n.$$

Since $\qquad a_1, a_2, a_3 \dots\dots a_n$ are in H. P.,

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \dots\dots \frac{1}{a_n} \text{ are in A. P.,}$$

and therefore
$$\frac{1}{a_1} - \frac{1}{a_2} = \frac{1}{a_2} - \frac{1}{a_3},$$

or
$$a_2 = \frac{2a_1a_3}{a_1 + a_3},$$

or
$$a_1a_2 + a_2a_3 = 2a_1a_3.$$

Hence we see that the law holds good when $n=3$.

Assume that it holds good for $n-1$ terms; that is,

$$a_1a_2 + a_2a_3 + \dots\dots a_{n-1}a_n = (n-1)a_1a_n.$$

To each side add a_na_{n+1},

$$\therefore\ a_1a_2 + a_2a_3 + \dots\dots a_{n-1}a_n + a_na_{n+1} = (n-1)(a_1a_n) + a_na_{n+1}.$$

But
$$\frac{1}{a_{n+1}} - \frac{1}{a_1} = n\left(\frac{1}{a_{n+1}} - \frac{1}{a_n}\right),$$

or
$$\frac{a_1 - a_{n+1}}{a_1a_{n+1}} = \frac{n(a_n - a_{n+1})}{a_na_{n+1}}.$$

Simplifying the result we obtain

$$(n-1)a_1a_n + a_na_{n+1} = na_1a_{n+1}.$$

$$\therefore\ a_1a_2 + a_2a_3 + \dots\dots a_na_{n+1} = na_1a_{n+1}.$$

Therefore if the law holds good for $n-1$ terms, it also holds good for n terms. But we have proved that it holds good for *two* terms, or when $n=3$; therefore it holds good for *three* terms, and therefore for any number of terms.

NOTE.—There is a more direct and simple proof of this problem; but it is here given as a good illustration of the application of the method of Induction.

EXERCISE XXVII.

Prove by Induction:

1. $1^2+2^2+3^2+4^2+\ldots\ldots n^2=\dfrac{n(1+n)(2n+1)}{6}$.

2. $1^3+2^3+3^3+4^3+\ldots\ldots n^3=\left(\dfrac{n(n+1)}{2}\right)^2$.

3. $1\,.\,2+2\,.\,3+3\,.\,4+4\,.\,5+\ldots\ldots n(n+1)=\dfrac{n(n+1)(n+2)}{3}$.

4. x^n+y^n is divisible by $x+y$ when n is an *odd* integer, and that x^n-y^n is divisible by $x+y$ when n is an *even* integer.

5. A series, a_1, b_1, a_2, $b_2\ldots$, is found according to the following law: a_n is an arithmetic mean between a_1 and b_{n-1}, and b_n is an harmonic mean between b_1 and a_{n-1}. Show that $a_nb_n=a_1b_1$.

6. If a_n, b_n be the coefficients of x^n in the expansions of

$$\frac{2-x}{1-4x+x^2} \text{ and } \frac{1}{1-4x+x^2}$$

respectively, then $a_n^2-3b_n^2=1$.

7. $2+6+14+30+\ldots\ldots$ to n terms $=2^{n+2}-(2n+4)$.

8. $(x+a)(x+b)(x+c)\ldots\ldots$ to n factors

$$=x^n+x^{n-1}(a+b+c+d+\ldots\ldots)$$
$$+x^{n-2}(ab+bc+ca+\ldots\ldots)$$
$$+x^{n-3}(abc+bcd+cda+\ldots\ldots)+\ldots\ldots abcd\ldots\ldots$$

Apply Induction to Examples 9–11:

9. Find the sum of n terms of the series, $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots.$

10. Find the sum of $2^2+4^2+6^2+\ \ldots$ to n terms.

11. Find the sum of $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}\ldots.$ to n terms

12. If $\dfrac{a+(a+y)x+(a+2y)x^2+(a+3y)x^3\ldots.\text{ to }\infty}{a+(a-y)x+(a-2y)x^2+(a-3y)x^3\ldots.\text{ to }\infty}=b$, and if x receive values in H. P., show that the corresponding values of y will be in A. P.

13. S_1, S_2, $S_3\ldots.$ are the sums to n terms of n geometric series, whose first terms are each unity, and common ratios 1, 2, 3…. Show that

$$S_1+S_2+2S_3+3S_4+\ldots.(n-1)S_n=1^n+2^n+3^n+\ldots.n^n.$$

14. $\dfrac{a+bx}{a-bx}=\dfrac{b+cx}{b-cx}=\dfrac{c+dx}{c-dx}=\ldots.$, and if, also, $a^x=b^y=c^z=\ldots.$, then will a, b, $c\ldots.$ be in G. P., and x, y, $z\ldots.$ in H. P.

15. A person devised his estate among n persons in the following manner: A was to receive \$$P$ and $\frac{1}{n}$ of the remainder; B, \$$2P$ and $\frac{1}{n}$ of the remainder; C, \$$3P$ and $\frac{1}{n}$ of the remainder, and so on. Find the value of the estate.

16. If the difference between the $(n-1)^{\text{th}}$ and n^{th} terms of an H. P. be $\dfrac{1}{an^2+bn^2+c}$, find the relation between a, b and c.

17. If $x=1+\frac{1}{n}$, show that the sum of the series,

$$1+2x+3x^2+\ldots.,$$

to n terms $=n^2$.

18. Show that if $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$, then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

19. If $x+\frac{1}{y}=y+\frac{1}{z}=z+\frac{1}{x}$, then $x^2y^2z^2=1$, or $x=y=z$.

20. Show that if

$x^2=y^2+z^2+2ayz$, $y^2=z^2+x^2+2bzx$ and $z^2=x^2+y^2+2cxy$, then

$$\frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}=\frac{z^2}{1-c^2}.$$

21. Show that if $\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}=1$, then

$$\left(\frac{b^2+c^2-a^2}{2bc}\right)^{2n+1}+\text{anal.}+\text{anal.}=1.$$

22. If
$$a^2x^2+b^2y^2+c^2z^2=0,$$
$$a^2x^3+b^2y^3+c^2x^3=0$$

and
$$\frac{1}{x}-a^2=\frac{1}{y}-b^2=\frac{1}{z}-c^2,$$

prove that
$$a^4x^3+b^4y^3+c^4z^3=0$$

and
$$a^6x^3+b^6y^3+c^6z^3=a^4x^2+b^4y^2+c^4z^2.$$

23. If x, y be unequal, and if $\frac{(2x-y-z)^3}{x}=\frac{(2y-z-x)^3}{y}$, then will each fraction $=\frac{(2z-x-y)^3}{z}$.

24. If $n(a^2+b^2+c^2+\ldots)=(a+b+c+\ldots)^2$, then $a=b=c=\ldots$, n being the number of the letters.

25. Show that $(x^2+xy+y^2)(a^2+ab+b^2)$ can be expressed in the form X^2+XY+Y^2.

CHAPTER XIX.

BINOMIAL THEOREM.

POSITIVE INTEGRAL EXPONENT.

257. By trial it can easily be shown that

$$(x+a)(x+b)(x+c) = x^3 + x^2(a+b+c) + x(ab+bc+ca) + abc;$$

also, $(x+a)(x+b)(x+c)(x+d)$

$$= x^4 + x^3(a+b+c+d) + x^2(ab+bc+cd+da+ac+bd)$$
$$+ x(abc+bcd+cda+dab) + abcd.$$

From these two cases it is apparent that the coefficient of the highest power of x is 1; the coefficient of the next highest power, the combinations of the *second* terms of the factors taken *one* at a time; the coefficient of the next highest power, the combinations of the *second* terms of the factors taken *two* at a time, and so on, the last term being the combinations of the terms taken all together.

258. The truth of this law of expansion for any number of factors may be proved by Induction; but it can readily be shown to be true by calling attention to the manner in which the various terms of the product are obtained. For instance, when $(x+a)(x+b)(x+c)(x+d)$ are actually multiplied together, the product is obtained according to the following laws:

1. The whole product consists of a number of partial products, obtained by multiplying together four letters, *one* being taken from each of the four factors.

2. The partial product x^4 is obtained by taking x out of each of the four factors.

3. The next term is obtained by taking x out of any *three* of the factors as many ways as possible, and *one* of the letters a, b, c, d out of the remaining factors.

4. The next term is obtained by taking x out of any *two* of the factors as many ways as possible, and two of the letters a, b, c, d out of the remaining two factors.

5. The next term is obtained by taking x out of any *one* of the factors as many ways as possible, and three of the letters a, b, c, d out of the remaining *three* factors.

6. The last term—the one independent of x—is obtained by taking all the letters a, b, c, d.

Now it is evident that this is only another way of stating that the coefficient of x^3 is the combinations of a, b, c, d taken *one* at a time; the coefficient of x^2, the combinations of a, b, c, d taken *two* at a time, and so on.

It is also evident that the same reasoning applies, no matter what the number of factors is, so long as each factor begins with the same letter (x). Consequently we reach the general result that

$$\begin{aligned}(x+a_1)(x+a_2)(x+a_3)\ldots(x+a_n)\\ &= x^n + x^{n-1}(a_1+a_2+\ldots a_n)\\ &\quad + x^{n-2}(a_1a_2+a_2a_3+a_3a_1+\ldots a_{n-1}a_n)+\ldots a_1a_2\ldots a_n,\end{aligned}$$

the coefficients being formed according to the law of combinations; that is, the coefficient of x^{n-1} is the combinations, taken *one* at a time, of a_1, a_2, $a_3 \ldots . a_n$; the coefficients of x^{n-2}, the combinations of the same letters taken *two* at a time, and so on. If, now, we assume $a_1 = a_2 = a_3 = \ldots . a_n = a$, we have

$$\begin{aligned}&(x+a)^n\\ &= x^n + x^{n-1}(na) + x^{n-2}\left\{\frac{n(n-1)}{\underline{|2}}a^2\right\} + x^{n-3}\left\{\frac{n(n-1)(n-2)}{\underline{|3}}a^3\right\} + \ldots a^n\\ &= x^n + nx^{n-1}a + \frac{n(n-1)}{\underline{|2}}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^{n-3}a^3 + \ldots . a^n.\end{aligned}$$

This is the **Binomial Theorem**; that is, the law of the expansion of an expression of *two* terms when the index is a positive integer. In a subsequent chapter it will be shown that the same law of expansion holds for any index.

259. *Proof of the Binomial Theorem.*—The previous result may be reached with much less work as follows:

$(x+a)^n$ is the product of n factors, each equal to $(x+a)$. This product consists of terms, each of n dimensions, obtained by multiplying together n letters, *one* being taken from each factor. For instance, the term $x^{n-3}a^3$ is obtained by taking x out of $(n-3)$ factors, and a out of the remaining *three*. Therefore the coefficient of $x^{n-3}a^3$, that is, the number of terms containing $x^{n-3}a^3$, will be the number of ways $(n-3)$ things can be taken out of n things, or the number of ways *three* things can be taken out of n things; that is,

$$\text{the coefficient of } x^{n-3}a^3 = \frac{n(n-1)(n-2)}{\underline{|3}} = \frac{\underline{|n}}{\underline{|n-3}\,\underline{|3}}.$$

Similarly it may be shown that

$$\text{the coefficient of } x^{n-r}a^r = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}.$$

Now, by giving r all values possible in this case, that is, 0, 1, 2, 3 .... n, we obtain the coefficients of all the terms. Therefore

$$(x+a)^n = x^n + C_1x^{n-1}a + C_2x^{n-2}a^2 + \dots C_na^n$$

where C_1, C_2, $C_3 \dots C_n$ represent the number of combinations of n things taken 1, 2, 3 .... n together.

Cor. 1.—Write $-a$ for a; then

$$(x-a)^n = x^n - C_1x^{n-1}a + C_2x^{n-2}a^2 - \dots + (-1)^na^n.$$

Cor. 2.—Since the coefficients of the expansion are

$$1,\ C_1,\ C_2,\ C_3 \dots C_n,$$

the number of terms in the expansion of $(x+a)^n$ is $(n+1)$.

Cor. 3.—Let $x=1$ and $a=x$; then

the expansion of $(1+x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots . C_nx^n$.

Cor. 4.—In the preceding let $x=1$; then

$$(1+1)^n = 2^n = 1 + C_1 + C_2 + C_3 + \ldots . C_n,$$

or

$$2^n - 1 = \quad C_1 + C_2 + C_3 + \ldots . C_n.$$

Thus by the Binomial Theorem we reach the conclusion already obtained in the chapter on Combinations, that the number of combinations of n things taken 1, 2, 3 .... n together $= 2^n - 1$.

Cor. 5.—In $(1+x)^n = 1 + C_1x + C_2x^2 + \ldots . C_nx^n$ put $x = -1$;

then $$(1-1)^n = 0 = 1 - C_1 + C_2 - C_3 + \ldots . (-1)^nC_n,$$

or $$C_1 + C_3 + C_5 + \ldots . = 1 + C_2 + C_4 + C_6 + \ldots .$$

That is, the sum of the combinations taken 1, 3, 5 .... together = the sum of the combinations taken 2, 4, 6 .... together, plus unity.

260. Any binomial can be expanded by using the form $(1+x)^n$. For, suppose we have to expand $(x+y)^n$, this can be expressed in the form,

$$\left\{x\left(1+\frac{y}{x}\right)\right\}^n = x^n\left(1+\frac{y}{x}\right)^n.$$

Let $\dfrac{y}{x} = a$; then $$(x+y)^n = x^n(1+a)^n.$$

We can now expand $(1+a)^n$ and multiply each term of the expansion by x^n.

261. Since the coefficients of $(a+x)^n$ are, after the first term, the combinations of n things taken 1, 2, 3 .... n together,

the coefficient of x is $\dfrac{|\underline{n}}{|\underline{1}\;|\underline{n-1}}a^{n-1}$; of x^2, $\dfrac{|\underline{n}}{|\underline{2}\;|\underline{n-2}}a^{n-2}$;

of x^3, $\dfrac{|\underline{n}}{|\underline{3}\;|\underline{n-3}}a^{n-3}$ .... ; and of x^r, $\dfrac{|\underline{n}}{|\underline{r}\;|\underline{n-r}}a^{n-r}$.

Therefore the term involving x^r in $(a+x)^n$ is

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} a^{n-r} x^r.$$

This is called the **General Term,** or the $(r+1)^{\text{th}}$ term of the series.

Cor.—The general term of $(1+x)^n$ is

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} x^r.$$

262. *The coefficients of $(x+a)^n$ equidistant from the beginning and end of the expansion are equal.*

For the coefficient of $x^{n-r}a^r$ is the number of combinations of n things, r together, and therefore equal to

$$\frac{\lfloor n}{\lfloor r \lfloor n-r};$$

it is also the $(r+1)^{\text{th}}$ coefficient from the *beginning* of the expansion. The coefficient of $x^r a^{n-r}$ is the number of combinations of n things, $(n-r)$ together, and therefore equal to

$$\frac{\lfloor n}{\lfloor n-r \lfloor r}.$$

But it is the $(r+1)^{\text{th}}$ coefficient from the *end* of the series; therefore, since

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} = \frac{\lfloor n}{\lfloor n-r \lfloor r},$$

the $(r+1)^{\text{th}}$ coefficient from the beginning $=$ the $(r+1)^{\text{th}}$ coefficient from the end.

The important point to notice in this almost self-evident proposition is that, since the combinations of n things, r together, $=$ combinations $(n-r)$ together, it follows that the coefficients must be equal when they are respectively the combinations of n things,

r together, and n things, $(n-r)$ together. This occurs when the terms are the $(r+1)^{\text{th}}$ and the $(n-r+1)^{\text{th}}$ from the beginning, or the $(r+1)^{\text{th}}$ from the beginning and the $(r+1)^{\text{th}}$ from the end—the $(r+1)^{\text{th}}$ from the end being the same term as the $(n-r+1)^{\text{th}}$ from the beginning, the whole number of terms being $(n+1)$.

Ex. 1.—Find the product of $(x+1)(x+2)(x+3)(x+4)$.

The first term is x^4;
the coefficient of $x^3 = (1+2+3+4)$;
the coefficient of $x^2 = (1.2+1.3+1.4+2.3+2.4+3.4)$;
the coefficient of $x \;= (1.2.3+2.3.4+3.4.1+4.1.2)$;
and the last term $= 1.2.3.4$.
$\therefore$ product $= x^4+10x^3+35x^2+50x+24$.

Ex. 2.—Write down the coefficient of x^2 in

$$(x-1)(x+2)(x+5)(x-6).$$

The coefficient

$$=\{(-1)(2)+(-1)(5)+(-1)(-6)+(2)(+5)+(2)(-6)+(5)(-6)\}$$
$$=\{-2-5+6+10-12-30\}=-33.$$

Ex. 3.—Expand $(x+y)^6$.

$$(x+y)^6 = x^6 + \frac{\underline{|6}}{\underline{|1}\,\underline{|5}}x^5y + \frac{\underline{|6}}{\underline{|2}\,\underline{|4}}x^4y^2 + \frac{\underline{|6}}{\underline{|3}\,\underline{|3}}x^3y^3 + \frac{\underline{|6}}{\underline{|4}\,\underline{|2}}x^2y^4$$
$$+\frac{\underline{|6}}{\underline{|5}\,\underline{|1}}xy^5+y^6$$
$$= x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6.$$

This expansion might have been written as follows:

$$x^6+\frac{6}{1}\,.\,x^5y+\frac{6.5}{1.2}\,.\,x^4y^2+\frac{6.5.4}{1.2.3}\,.\,x^3y^3+\frac{6.5.4.3}{1.2.3.4}\,.\,x^2y^4$$
$$+\frac{6.5.4.3.2}{1.2.3.4.5}\,.\,xy^5+y^6.$$

N.B.—The student will observe that after the middle term, $\frac{6.5.4}{1.2.3}.x^3y^3$, is passed, the coefficients are the same as in the first half of the series, only in reverse order. In such examples, therefore, the coefficients of the last half of the series can at once be written down by observing the coefficients of the first half. This is, of course, only an application of the general law that the coefficients equidistant from beginning and end are equal.

Ex. 4.—Find the coefficient of x^4 in the expansion of $(3+2x)^8$.

The coefficient of x^r in the expansion of $(a+x)^n$ is

$$\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}a^{n-r}=\frac{n(n-1)(n-2)(n-3)\ldots(n-r+1)a^{n-r}}{1.2.3\ldots r}.$$

In this example $a=3$ and $x=2x$; $\therefore$ the coefficient of x^4 is

$$\frac{\underline{|8}}{\underline{|4}\,\underline{|4}}.3^4.2^4=\frac{8.7.6.5}{1.2.3.4}\times 3^4\times 2^4.$$

N.B.—Instead of deducing the result from the general expression, $(a+x)^n$, it is advisable for the student beginning the subject to obtain the term required by actual expansion.

Ex. 5.—Find the coefficient of x^6 in the expansion of $\left(x+\frac{1}{x}\right)^{10}$.

$$x+\frac{1}{x}=x\left(1+\frac{1}{x^2}\right);\quad\therefore\quad\left(x+\frac{1}{x}\right)^{10}=x^{10}\left(1+\frac{1}{x^2}\right)^{10}.$$

We have now to expand $\left(1+\frac{1}{x^2}\right)^{10}$ and then multiply each term of the expression by x^{10}. To obtain x^6 we must find the term in $\left(1+\frac{1}{x^2}\right)^{10}$, which has x^4 for a denominator (since $x^{10}\div x^4=x^6$). Expanding $\left(1+\frac{1}{x^2}\right)^{10}$ we find the third term has x^4 in the denominator, and its coefficient is $\frac{10\times 9}{1.2}=45$, which is the coefficient required.

Another way of arranging $\left(x+\frac{1}{x}\right)^{10}$ is as follows:

$$x+\frac{1}{x}=\frac{1+x^2}{x},\quad\therefore\quad\left(x+\frac{1}{x}\right)^{10}=\frac{(1+x^2)^{10}}{x^{10}}.$$

In this case, to find x^6 we must find the term in the numerator which contains x^{16}; and this is found to be the third term from the end, which has the same coefficient as the third term from the beginning, and, as before, is found to be 45.

Ex. 6.—Find by the Binomial Theorem the value of

$$(a+\sqrt{a^2-1})^6+(a-\sqrt{a^2-1})^6.$$

Let $\sqrt{a^2-1}=x$,

$$\therefore\ (a+\sqrt{a^2-1})^6+(a-\sqrt{a^2-1})^6=(a+x)^6+(a-x)^6.$$

$$(a+x)^6=a^6+6a^5x+15a^4x^3+20a^2x^3+15a^2x^4+6ax^5+x^6,$$

and $\quad (a-x)^6=a^6-6a^5x+15a^4x^3-20a^3x^3+15a^2x^4-6ax^5+x^6.$

Adding, $(a+x)^6+(a-x)^6$

$$=2(a^6+15a^4x^2+15a^2x^4-x^6)$$
$$=2\{a^6+15a^4(a^2-1)+15a^2(a^2-1)^2+(a^2-1)^3\}$$
$$=2\{a^6+15a^6-15a^4+15a^6-30a^4+15a^2+a^6-3a^4+3a^2-1\}$$
$$=2\{32a^6-48a^4+18a^2-1\}.$$

It will be observed in this example that owing to the second terms of each binomial differing in sign alone, the *even* terms of the expansions disappear when they are added, and the *odd* terms are taken twice. If the difference of expansions had been required, the *odd* terms would have disappeared, and the *even* terms would be each taken twice. It is, therefore, not necessary to write out the expansions in full—the result can be written down at once by inspection.

EXERCISE XXVIII.

Write out the following expansions:

1. $(x+a)(x+b)(x-c)$. 2. $(x-5)(x+4)(x-3)(x-6)$.

3. $(x-6)(x+18)(x-9)(x-3)$.

4. $(x+a)^5$.

5. $(2a+3b)^6$.

6. $(2a-y)^4$.

7. $(1-2x)^6$.

8. $\left(x+\dfrac{2}{x}\right)^6$.

9. $\left(x^2+\dfrac{1}{x^2}\right)^4$.

10. Find the third term of $(x-3y)^8$.

11. Find the fourth term of $(2x-5)^6$.

12. Find the twentieth term of $(a+b)^{25}$.

13. Find the thirty-fifth term of $(4x-a)^{40}$.

14. Find the middle term of $(1+x)^{10}$.

15. Find the middle term of $(x-2y)^{12}$.

16. Find the middle term of $(1-x)^{12}$.

17. Find the general term of $(x-3y)^n$.

18. Find the general term of $(x^2+y^3)^n$.

19. Find the general term of $(x^2-y^3)^{16}$.

20. Show that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is double the coefficient of x^n in the expansion of $(1+x)^{2n-1}$.

21. Find the middle term of $(1+x)^{2n}$, and prove it

$$=\frac{1\,.\,3\,.\,5\,.\,\ldots\,.\,(2n-1)}{\lfloor n}\,.\,2^n x^n.$$

22. Find the coefficient of x^r in the expansion of $\left(x+\dfrac{1}{x}\right)^n$.

23. Find the middle term of $\left(x-\dfrac{1}{x}\right)^{2n}$.

24. The coefficients of the fifth, sixth and seventh terms of the expansion of $(1+x)^{2n}$ are in A. P. Find n.

25. For what values of n are the coefficients of the second, third and fourth terms of the expansion of $(1+x)^n$ in A. P.?

26. Simplify $(x+\sqrt{y^2-1})^4+(x-\sqrt{y^2-1})^4$.

27. Simplify $(\sqrt{m^2+1}+\sqrt{m^2-1})^6-(\sqrt{m^2+1}-\sqrt{m^2-1})^6$.

28. If a be the sum of the odd terms, and b the sum of the even terms, of $(1+x)^n$, show that $(1-x^2)^n=a^2-b^2$.

29. Simplify $(5\sqrt{2}+7)^n\times(5\sqrt{2}-7)^n$.

30. Find the term independent of x in $\left(\frac{2}{3}x^2-\frac{1}{4x}\right)^6$.

31. Find the term independent of x in $\left(x-\frac{1}{x^2}\right)^{2n}$.

32. Find the coefficient of x^r in the expansion of $\left(x^2+\frac{1}{x^3}\right)^n$.

33. Find the coefficient of x^m in the expansion of $\left(x^3+\frac{1}{x^2}\right)^{2n}$.

34. Find the value of

$$1+2n+\frac{3n(n-1)}{\underline{|2}}+\frac{4n(n-1)(n-2)}{\underline{|3}}+\dots.(n+1)1$$

when n is a positive integer.

263. *To find the greatest coefficient in the expansion of $(1+x)^n$ when n is a positive integer.*

This is the same problem as that of finding the value of r for which the number of combinations of n things, taken r together, will be the greatest. In Art. 244 it was proved that when n is even the value of r is $\frac{n}{2}$, and when n is odd the value of r is $\frac{n-1}{2}$ or $\frac{n+1}{2}$. Therefore the greatest coefficient when n is even is the coefficient of the $\left(\frac{n}{2}+1\right)^{\text{th}}$ term, and when n is odd, the $\left(\frac{n-1}{2}+1\right)^{\text{th}}$ or $\left(\frac{n+1}{2}+1\right)^{\text{th}}$ term; that is, of the $\left(\frac{n+1}{2}\right)^{\text{th}}$ or $\left(\frac{n+3}{2}\right)^{\text{th}}$ term.

264. *To find the numerically greatest term in the expansion of* $(a+x)^n$.

The r^{th} term of $(a+x)^n$ is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{\lfloor r-1}a^{n-r+1}\,.\,x^{r-1},$$

and the $(r+1)^{\text{th}}$ term is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\lfloor r}a^{n-r}\,.\,x^{r};$$

that is, the $(r+1)^{\text{th}}$ term $= r^{\text{th}}$ term $\times \dfrac{n-r+1}{r}\cdot\dfrac{x}{a}$.

Therefore the $(r+1)^{\text{th}}$ term $>, =, <, r^{\text{th}}$ term

according as $\dfrac{n-r+1}{r}\cdot\dfrac{x}{a} >, =, <, 1;$

or as $$\frac{n-r+1}{r} >, =, <, \frac{a}{x};$$

or as $$\frac{n+1}{r}-1 >, =, <, \frac{a}{x};$$

or as $$\frac{n+1}{r} >, =, <, 1+\frac{a}{x};$$

or as $$\frac{n+1}{1+\dfrac{a}{x}} >, =, <, r.$$

If $\dfrac{n+1}{1+\dfrac{a}{x}}$ is an integer, the $(r+1)^{\text{th}}$ term $=$ the r^{th} term when

$$r=\frac{n+1}{1+\dfrac{a}{x}},$$

and these will be the greatest terms of the expansion; for any greater value of r will make

$$\frac{n-r+1}{r}\cdot\frac{x}{a}<1.$$

If $\dfrac{n+1}{1+\dfrac{a}{x}}$ is not an integer, but has for its integral part m, then

$$\frac{n-r+1}{r} \cdot \frac{x}{a}$$

cannot be > 1 for any value of $r > m$; that is, the $(r+1)^{\text{th}}$ term, cannot be $> r^{\text{th}}$ term for any value of $r > m$. Therefore the $(r+1)^{\text{th}}$ term $> r^{\text{th}}$ until $r = m$; $\therefore$ the greatest term is the $(m+1)^{\text{th}}$ term.

N.B.—The student should observe that this proof applies solely to the numerically greatest term, and therefore applies to $(a-x)^n$.

Ex. 1.—Find the greatest term in the expansion of $(1+x)^n$ when

$$x = \frac{2}{3} \text{ and } n = 6.$$

The $(r+1)^{\text{th}}$ term is $>, =, <, r^{\text{th}}$ term,

according as $$\frac{n-r+1}{r} . x >, =, <, 1;$$

or as $$\frac{n+1}{1+\dfrac{1}{x}} >, =, <, r.$$

But $$n = 6 \text{ and } x = \frac{2}{3}.$$

$\therefore$ according as $$\frac{7}{1+\dfrac{3}{2}} >, =, <, r;$$

or as $$\frac{14}{5} >, =, <, r.$$

The greatest value r can have in order that $\frac{14}{5}$ may be $> r$ is 2; therefore the third term is the greatest, and its value is

$$\frac{6 \times 5}{1.2} \cdot \left(\frac{2}{3}\right)^2 = 15 \times \frac{4}{9} = 6\tfrac{2}{3}.$$

Ex. 2.—Find the greatest term in the expansion of $(a+x)^n$ when

$$a=\frac{1}{3}, \quad x=\frac{1}{4}, \quad n=8.$$

In this the condition
$$\frac{n+1}{1+\frac{a}{x}} >, =, <, r,$$

becomes
$$\frac{9}{1+\frac{4}{3}} >, =, <, r,$$

or
$$\frac{27}{7} >, =, <, r.$$

Therefore the greatest value of r is 3, and the fourth term is the greatest, and its value is,

$$\frac{8\times 7\times 6}{1\times 2\times 3}\left(\frac{1}{3}\right)^5\left(\frac{1}{4}\right)^3 = 56\times\frac{1}{243}\times\frac{1}{64}.$$

265. The following examples are suggestive, and worthy of attention:

Ex. 1.—Find the sum of the series,

$$1^2+\left(\frac{n}{\lfloor 1}\right)^2+\left\{\frac{n(n-1)}{\lfloor 2}\right\}^2+\ldots\ldots 1^2;$$

that is, find the sum of the squares of the coefficients of $(1+x)^n$.

$$(1+x)^n = 1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\ldots\ldots x^n; \qquad (1)$$

also,
$$(x+1)^n = x^n+nx^{n-1}+\frac{n(n-1)}{\lfloor 2}x^{n-2}+\ldots\ldots 1. \qquad (2)$$

If, now, (1) and (2) be multiplied together, and the coefficients of x^n be collected, their sum will be the given series.

But $$(1) \times (2) = (1+x)^n \times (x+1)^n = (1+x)^{2n}.$$

Therefore the given series must be equal to the coefficient of x^n in $(1+x)^{2n}$; *i.e.*, equal to

$$\frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}.$$

Hence $$1 + \left(\frac{n}{\underline{|1}}\right)^2 + \left\{\frac{n(n-1)}{\underline{|2}}\right\}^2 + \ldots\ldots 1^2 = \frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}.$$

Ex. 2.—If C_0, C_1, $C_2 \ldots\ldots C_n$ denote the coefficients of $(1+x)^n$, find the sum of

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_n}{n+1}.$$

Now, $$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots = S = 1 + \frac{n}{\underline{|2}} + \frac{n(n-1)}{\underline{|3}} + \ldots\ldots$$

Multiply both sides by $(n+1)$;

$$\therefore\ S(n+1) = (n+1) + \frac{(n+1)n}{\underline{|2}} + \frac{(n+1)(n)(n-1)}{\underline{|3}} + \ldots\ldots 1.$$

Add 1 to both sides;

$$\therefore\ S(n+1) + 1 = 1 + (n+1) + \frac{(n+1)(n)}{\underline{|2}} + \ldots\ldots 1 = (1+1)^{n+1} = 2^{n+1}.$$

$$\therefore\ S = 2^{n+1} - 1 \div (n+1).$$

EXERCISE XXIX.

Find the greatest term in the expansion of:

1. $(a+b)^{20}$, when $a = 2$, $b = 3$.
2. $(2x - y)^{12}$, when $x = 4$, $y = 5$.
3. $\left(x + \frac{y}{2}\right)^8$, when $x = 6$, $y = 8$.
4. $\left(\frac{2x}{3} + \frac{3a}{4}\right)^6$, when $x = 9$, $a = 16$.

Find the value of the greatest term in the following:

5. $(1+x)^n$, when $x=\frac{1}{3}$, $n=4$.

6. $(2-3x)^n$, when $x=\frac{1}{2}$, $n=6$.

7. $(a-b)^{2r}$, when $a=2$, $b=3$, $r=4$.

8. $\left(x+\frac{1}{x}\right)^n$, when $x=2$, $n=8$.

9. In the expansion of $(1+x)^{43}$ the coefficients of the $(2r+1)^{\text{th}}$ and $(r+2)^{\text{th}}$ terms are equal. Find r.

10. The second, third and fourth terms of $(a+x)^n$ are 240, 720 and 1080 respectively. Find the values of x and n.

11. Find the relation between r and n in order that the coefficients of the fifth and $(2r+5)^{\text{th}}$ terms of $(1+x)^n$ may be equal.

12. If the coefficient of the $3r^{\text{th}}$ term from the beginning of $(1+x)^{2n}$ equals the coefficient of the $(r+2)^{\text{th}}$ term from the end, find the relation between r and n.

If $a_0, a_1, a_2 \dots . a_n$ denote the coefficients of $(1+x)^n$, prove:

13. $a_1+2a_2+3a_3+\dots . na_n=n \cdot 2^{n-1}$.

14. $a_0-\frac{1}{2}a_1+\frac{1}{3}a_2+\dots .(-1)^n\frac{a_n}{n+1}=\frac{1}{n+1}$.

15. $a_1-\frac{a_2}{2}+\frac{a_3}{3}-\dots .(-1)^{n-1}\frac{a_n}{n}=1+\frac{1}{2}+\frac{1}{3}+\dots .\frac{1}{n}$.

16. $a_0-2a_1+3a_2-\dots .(-1)^n(n+1)a_n=0$.

17. $\frac{a_1}{a_0}+\frac{2a_2}{a_1}+\frac{3a_3}{a_2}+\dots .\frac{na_n}{a_{n-1}}=\frac{n(n+1)}{2}$.

18. $(a_0+a_1)(a_1+a_2)\dots .(a_{n-1}+a_n)=\frac{a_1a_2\dots . a_n(n+1)^n}{\underline{|n}}$.

19. $a_1-2a_2+3a_3-\dots .(-1)^{n-1}na_n=0$.

20. $a_0+2a_1+3a_2+\dots .(n+1)a_n=(n+2)2^{n-1}$.

21. $a_2 + 2a_3 + 3a_4 + \ldots . (n-1)a_n = 1 + (n-2)2^{n-1}$.

22. $a_0 a_r + a_1 a_{r+1} + \ldots . a_{n-r} a_n = \dfrac{\lfloor 2n}{\lfloor n-r \, \lfloor n+r}$.

23. $a_0 a_1 + a_1 a_2 + \ldots . a_{n-1} a_n = \dfrac{\lfloor 2n}{\lfloor n+1 \, \lfloor n-1}$.

24. $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots . = (-1)^{\frac{n}{2}} \dfrac{\lfloor n}{\left(\lfloor \frac{n}{2}\right)^2}$ when n is even, and $= 0$ when n is odd.

CHAPTER XX.

BINOMIAL THEOREM.

ANY EXPONENT.

266. In the preceding chapter we found the *form* that the expansion of $(a+x)$ assumes when n is a positive integer. This was easily obtained, since $(a+x)^n$ was taken as the product of n equal factors, and therefore its coefficients came under the law of combinations.

We have now to prove that the *form* of the expansion of $(a+x)^n$ is the same when n is not a positive integer as when n is a positive integer.

267 By actual division we find that

$$\frac{1}{(1+x)^2} \text{ or } (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots.$$

$$= 1 + \frac{(-2)}{\underline{|1}}x + \frac{(-2)(-2-1)}{\underline{|2}}x^2 \ldots.$$

Hence we see that in this particular case the formula holds good. Similarly, by actually extracting the square root of $1+x$ it can be shown that

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{\underline{|2}}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{\underline{|3}}x^3 + \ldots.$$

268. We now proceed to prove that the law holds good for all values of n, fractional and negative. At the outset it will be

necessary to call the attention of the student particularly to the following statement, the truth of which he must be convinced before he can understand the proof of the Binomial Theorem for any exponent.

269. *If two algebraic expressions are multiplied together, the* FORM *of the product is* INDEPENDENT *of the value of the letters involved.*

Thus $(a + b)(a - b) = a^2 - b^2$, no matter what values may be given to a and b. So, too, if

$$(a_0 + a_1x + a_2x^2 + \ldots, a_nx^n)(b_0 + b_1x + b_2x^2 + \ldots . b^nx^n)$$
$$= A_0 + A_1x + A_2x^2 + \ldots . A_{2n}x^{2n},$$

the form of the product is independent of the values of a_0, b_0, a_1, $b_1 \ldots .$ Of course, the *value* of the product will change, but its algebraic expression will remain the same. The application of this to the proof of the Binomial will be seen in the following Art.

270. *To prove the Binomial Theorem when the exponent is a positive fraction.*

Let the series,

$$1 + mx + \frac{m(m-1)}{\underline{|2}}x^2 + \ldots, \qquad (1)$$

be denoted by $f(m)$; then the series,

$$1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \ldots, \qquad (2)$$

will be denoted by $f(n)$.

By the reasoning of the preceding Art. the form of the product of (1) and (2) will be the same, whatever values be given to m and n. But the series (1) is the expansion of $(1 + x)^m$ when m

is a positive integer, and (2) is the expansion of $(1+x)^n$ when n is a positive integer. Therefore the product of (1) and (2), when m and n are positive integers, is

$$(1+x)^{m+n} \text{ or } 1+(m+n)x+\frac{(m+n)(m+n-1)}{\underline{|2}}x^2+\ldots. \quad (3)$$

Hence (3) is the form of the product of $f(m)$ and $f(n)$ for all values of m and n. Also, since $f(m)$ denotes (1), $f(m+n)$ denotes (3), and therefore

$$f(m)\times f(n)=f(m+n);$$

also,
$$f(m)\times f(n)\times f(p)=f(m+n)\times f(p)$$
$$=f(m+n+p).$$

Similarly it can be shown that

$$f(m)\times f(n)\times f(p)\ldots. \text{ to } n \text{ factors}=f(m+n+p\ldots. \text{ to } n \text{ terms}).$$

Now, let
$$m=n=p=\ldots.\frac{h}{k},$$

where h and k are positive integers.

$$\therefore f\left(\frac{h}{k}\right)\times f\left(\frac{h}{k}\right)\ldots. \text{ to } k \text{ factors}=f\left(\frac{h}{k}+\frac{h}{k}\ldots. \text{ to } k \text{ terms}\right),$$

or
$$\left\{f\left(\frac{h}{k}\right)\right\}^k=f(h)=(1+x)^h, \quad \left\{\begin{matrix}\text{since } h \text{ is a posi-}\\ \text{tive integer.}\end{matrix}\right\}$$

or
$$f\left(\frac{h}{k}\right)=(1+x)^{\frac{h}{k}}.$$

But, in accordance with the notation employed,

$$f\left(\frac{h}{k}\right)=1+\frac{\frac{h}{k}\cdot x}{\underline{|1}}+\frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{\underline{|2}}x^2+\ldots.$$

$$\therefore (1+x)^{\frac{h}{k}}=1+\frac{\frac{h}{k}\cdot x}{\underline{|1}}+\frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{\underline{|2}}x^2+\ldots$$

Let $\frac{h}{k} = n$; then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \ldots\ldots;$$

that is, the Binomial Theorem holds good when n is a positive fraction, $\left(\frac{h}{k}\right)$.

271. *To prove the Binomial Theorem for a negative index.*

Since $f(m) \times f(n) = f(m+n)$ for all values of m and n, let $n = -m$ (m being taken positive).

Then $$f(m) \times f(-m) = f(m-m) = f(0).$$

But $f(0) = 1$, as it is obtained by putting $m = 0$ in the series,

$$1 + mx + \frac{m(m-1)}{\lfloor 2}x^2 + \ldots\ldots$$

$$\therefore\ f(m) \times f(-m) = 1,$$

or $$f(-m) = \frac{1}{f(m)} = \frac{1}{(1+x)^m} = (1+x)^{-m}.$$

But $$(1+x)^{-m} = 1 + \frac{-m}{\lfloor 1}x + \frac{(-m)(-m-1)}{\lfloor 2}x^2 + \ldots\ldots,$$

since $f(-m)$ stands for $1 + \frac{(-m)}{\lfloor 1}x + \frac{(-m)(-m-1)}{\lfloor 2}x^2 + \ldots\ldots$

Thus it is seen that the Binomial Theorem holds good for the *negative* index, $-m$.

272. The proof contained in the two preceding Arts. presents one difficulty, which needs some explanation. It has been stated that $f(m) \times f(n) = f(m+n)$. Now, what meaning must be attached to such a statement when the series which $f(m)$ and $f(n)$ represent are *divergent?* Such series are the expansions of $(1-x)^{-2}$ and $(1-x)^{-1}$ when $x \overline{\underline{>}} 1$. (See next Art.) When $x < 1$, $(1-x)^{-2}$ is

arithmetically $= 1 + 2x + 3x^2 + 4x^3 + \dots \infty$, and $(1-x)^{-1}$ is arithmetically $= 1 + x + x^2 + x^3 + \dots \infty$; and $\therefore$ $(1-x)^{-2} \times (1-x)^{-1}$ or $(1-x)^{-3}$ is arithmetically $= 1 + 3x + 6x^2 + \dots \infty$, which is the product of $(1 + 2x + 3x^2 + 4x^3 + \dots \infty)(1 + x + x^2 + \dots \infty)$. But when $x \overset{=}{>} 1$, $(1-x)^{-2}$ and $(1-x)^{-1}$ are not respectively arithmetically $= 1 + 2x + 3x^2 + 4x^3 + \dots \infty$ and $1 + x + x^2 + x^3 + \dots \infty$, and we cannot assert that $(1 + 2x + 3x^2 + \dots \infty) \times (1 + x + x^2 + x^3 + \dots \infty) = 1 + 3x + 6x^2 + \dots \infty$. We can, however, assert that the first r terms of the product of $(1 + 2x + 3x^2 + \dots \infty)$ and $(1 + x + x^2 + \dots \infty)$ are the same as the first r terms of $(1 + 3x + 6x^2 + \dots \infty)$, and, generally, that the first r terms of the $f(m) \times f(n)$ are the same as the first r terms of $f(m+n)$.

273. It has been stated in the preceding Art. that the expressions, $1 + 2x + 3x^2 + \dots \infty$ and $1 + x + x^2 + x^3 + \dots \infty$, are, when $x < 1$, the arithmetical equivalents of $(1-x)^{-2}$ and $(1-x)^{-1}$. This can be shown by summing the series according to methods already employed in the chapters on Progressions. If, however, $x \overset{=}{>} 1$, $1 + 2x + 3x^2 + \dots \infty$ and $1 + x + x^2 + \dots \infty$ are not the arithmetical equivalents of $(1-x)^{-2}$ and $(1-x)^{-1}$. This can be easily proved as follows:

For, if possible, let $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots \infty$. Then if these expressions are identities they must be equal when $x = 2$, in which case

$$(1-x)^{-2} = (1-2)^{-2} = (-1)^{-2} = \frac{1}{(-1)^2} = 1,$$

and $1 + 2x + 3x^2 + \dots \infty = 1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots \infty = \infty$.

That is, $1 = \infty$, which is absurd.

The real value of $(1-x)^{-2}$ when $x \overset{=}{>} 1$ is $\dfrac{1}{(1-x)^2}$, which, when actually divided out, gives

$$1 + 2x + 3x^2 + 4x^3 + \dots \frac{(r+1)x^r}{(1-x)^2}.$$

If $x<1$, by taking r great enough the expression,

$$\frac{(r+1)x^r}{(1-x)^2},$$

can be made as small as we like, since x^r tends to vanish when $x<1$ and r is very great. If, however, $x \overset{=}{>} 1$, x^r increases as r increases, and each term of the quotient becomes greater than the preceding.

274. In the expansion of $(1+x)^n$ it was found that the $(r+1)^{\text{th}}$ term was obtained from the r^{th} term by multiplying the r^{th} term by $\frac{n-r+1}{r}\cdot x$. Now, if n is a positive integer, $\frac{n-r+1}{r}x$ or $\frac{n-(r-1)}{r}\cdot x$ becomes zero when $r-1=n$ or $r=n+1$. Therefore the number of terms when n is a positive integer cannot exceed $(n+1)$. But when n is negative or fractional, $\frac{n-(r-1)}{r}$ can never become zero, since $(r-1)$ is a positive integer and n is not a positive integer. Therefore when n is negative or fractional, the series does not terminate, and the number of terms is infinite.

Ex. 1.—Expand $(1+x)^{\frac{3}{2}}$.

$$(1+x)^{\frac{3}{2}} = 1 + \frac{\frac{3}{2}\cdot x}{\lfloor 1} + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{\lfloor 2}x^2 + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{\lfloor 3}x^3 + \dots$$

$$= 1 + \frac{3}{\lfloor 1}\cdot\left(\frac{1}{2}\right)x + \frac{3\,.\,1}{\lfloor 2}\cdot\left(\frac{1}{2}\right)^2\cdot x^2 - \frac{3.1.1}{\lfloor 3}\cdot\left(\frac{1}{2}\right)^3\cdot x^3 + \dots$$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \dots.$$

Ex. 2.—Expand $(1+x)^{\frac{1}{3}}$.

$$(1+x)^{\frac{1}{3}} = 1 + \frac{\frac{1}{3}}{\lfloor 1}x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{\lfloor 2}x^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{\lfloor 3}x^3 + \ldots.$$

$$= 1 + \frac{x}{3\lfloor 1} - \frac{1.2}{3^2\lfloor 2}x^2 + \frac{1.2.5}{3^3\lfloor 3}x^3 - \frac{1.2.5.8}{3^4\lfloor 4}x^4 + \ldots.$$

Ex. 3.—Expand $(1-x)^{-2}$.

$$(1-x)^{-2} = 1 + \frac{(-2)}{\lfloor 1}(-x) + \frac{(-2)(-2-1)}{\lfloor 2}(-x)^2$$

$$+ \frac{(-2)(-2-1)(-2-2)}{\lfloor 3}(-x)^3 + \ldots.$$

$$= 1 + 2x + 3x^2 + 4x^3 + \ldots.$$

Similarly it can be shown that

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots.$$

and $$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots.$$

Ex. 4.—Expand $(1-x)^{-n}$.

$$(1-x)^{-n} = 1 + \frac{(-n)}{\lfloor 1}(-x) + \frac{(-n)(-n-1)}{\lfloor 2}(-x)^2$$

$$+ \frac{(-n)(-n-1)(-n-2)}{\lfloor 3}(-x)^3 + \ldots$$

$$= 1 + nx + \frac{n(n+1)}{\lfloor 2}x^2 + \frac{n(n+1)(n+2)}{\lfloor 3}x^3 + \ldots.$$

This expansion, as to form, does not depend upon the value of n, whether integral or fractional, so long as n is positive; therefore when the index is negative, and the sign between the two terms of the binomial is negative—the terms themselves being

positive—the expansion has every term positive. The recognition of this fact will save much time in expanding binomial expressions. It is also worthy of notice that the factors of the numerators increase by unity continuously, instead of diminishing.

Ex. 5.—Expand $(9+2x)^{\frac{1}{2}}$.

$$9+2x=9\left(1+\frac{2x}{9}\right);$$

$$\therefore\ (9+2x)^{\frac{1}{2}}=9^{\frac{1}{2}}\left(1+\frac{2x}{9}\right)^{\frac{1}{2}}=3\left(1+\frac{2x}{9}\right)^{\frac{1}{2}}$$

$$=3\left\{1+\frac{\frac{1}{2}}{\lfloor 1}\left(\frac{2x}{9}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{\lfloor 2}\cdot\left(\frac{2x}{9}\right)^2+\ldots\right\}.$$

The expressions in the brackets can be simplified, and the results multiplied by 3.

Ex. 6.—Find the general term in the expansion of $(1+x)^{\frac{2}{3}}$.

The general term in the expansion of $(1+x)^n$ is

$$\frac{n(n-1)(n-2)(n-3)\ldots(n-r+1)}{\lfloor r}x^r.$$

In this case $n=\dfrac{2}{3}$; therefore the general term is

$$\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)\ldots\left(\frac{2}{3}-r+1\right)}{\lfloor r}x^r$$

$$=\frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\ldots\left\{-\frac{(3r-5)}{3}\right\}}{\lfloor r}x^r$$

$$=\frac{2\cdot 1\cdot 4\ldots(3r-5)}{3^r\lfloor r}(-1)^{r-1}x^r.$$

Ex. 7.—Find the general term in the expansion of $(1-x)^{-n}$.

The general term in the expansion of $(1+x)^n$ is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\lfloor r}x^r;$$

and to find the general term in the expansion of $(1-x)^{-n}$ we must substitute $(-x)$ for x and $(-n)$ for n. Therefore the general term

$$=\frac{-n(-n-1)(-n-2)\ldots\ldots(-n-r+1)}{\lfloor r}(-x)^r$$

$$=\frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{\lfloor r}(-1)^r(-x)^r$$

$$=\frac{n(n+1)\ldots\ldots(n+r-1)}{\lfloor r}x^r.$$

N.B.—Ex. 4 is an illustration of this general case.

Ex. 8.—Find the simplest form of the general term of $(1-x)^{-n}$ when n is a positive integer.

The general term is

$$\frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{\lfloor r}x^r$$

$$=\frac{1.2.3\ldots\ldots(n-1)n(n+1)\ldots\ldots(n+r-1)}{1.2.3\ldots\ldots(n-1)\lfloor r}x^r$$

$$=\frac{\lfloor n+r-1}{\lfloor n-1\,\lfloor r}x^r.$$

If $n=4$, the general term of $(1-x)^{-4}$ is

$$\frac{\lfloor r+3}{\lfloor 3\,\lfloor r}x^r=\frac{(r+1)(r+2)(r+3)}{6}x^r.$$

EXERCISE XXX.

1. Expand $(1+x)^{\frac{3}{4}}$.
2. Expand $(1-x)^{\frac{2}{3}}$.
3. Expand $(1+2x)^{\frac{1}{2}}$.
4. Expand $(a+bx)^{\frac{1}{2}}$.
5. Expand $(2-5x)^{\frac{2}{5}}$.
6. Expand $(1+x)^{-3}$.
7. Expand $(1-x)^{-6}$.
8. Expand $(2-x)^{-3}$.
9. Expand $(3-2x)^{-\frac{2}{3}}$.
10. Expand $(1+x^2)^{-3}$.
11. Expand $\sqrt[3]{(a^3-x^3)^2}$.
12. Expand $\dfrac{1}{\sqrt{1+3x}}$.
13. Expand $\dfrac{1}{\sqrt[n]{a^n-nx}}$.
14. Find the sixth term of $(1+2x)^{-\frac{1}{2}}$.
15. Find the eighth term of $(1-2y^2)^{\frac{9}{2}}$.
16. Find the fifth term of $(1+3x^3)^{\frac{10}{3}}$.
17. Find the seventh term of $(a-x)^{\frac{5}{2}}$.
18. Find the tenth term of $(3-2b)^{\frac{7}{3}}$.
19. Find the $(r+1)^{th}$ term of $(1-x)^{-\frac{4}{3}}$.
20. Find the $(r+1)^{th}$ term of $(1+x)^{\frac{11}{3}}$.
21. Find the $(r+1)^{th}$ term of $(1-2y)^{-3}$.
22. Find the $(r+1)^{th}$ term of $(m-3n)^{-4}$.
23. Find the $(r+1)^{th}$ term of $(x+2y)^{-\frac{1}{2}}$.
24. Find the twelfth term of $(2^{10}-2^7x)^{\frac{13}{2}}$.
25. Find the sixth term of $(3^8+6^4x)^{\frac{11}{2}}$.
26. Find the eighth term of $(1+3x)^{\frac{3}{4}}$.
27. Find the third and sixth terms of $(1-\frac{1}{2}x)^{-\frac{3}{2}}$.

28. Find in its simplest form the $(r+1)^{\text{th}}$ term of $(1-x)^{-5}$.

29. Expand $\left(\frac{1}{x}-\frac{1}{x^2}\right)^{-4}$.

30. Find the coefficient of x^{11} in $(1-4x)^{-\frac{1}{2}}$.

275. *To find (when possible) the numerically greatest term in the expansion of* $(a+x)^n$.

The $(r+1)^{\text{th}}$ term of $(a+x)^n$ is obtained from the r^{th} term by multiplying it by $\frac{n-r+1}{r}\cdot\frac{x}{a}$, no matter what value may be given to n.

I. Let n be a positive integer.

This case has been discussed already in Art. 264.

II. Let n be positive and fractional.

Then, since $(r+1)^{\text{th}}$ term $= r^{\text{th}} \times \left(\frac{n-r+1}{r}\right)\frac{x}{a}$, the $(r+1)^{\text{th}}$ term $\begin{matrix}>\\=\\<\end{matrix}$ r^{th} term

according as $$\left(\frac{n-r+1}{r}\right)\cdot\frac{x}{a} \begin{matrix}>\\=\\<\end{matrix} 1,$$

or as $$\left(\frac{n+1}{r}-1\right) \begin{matrix}>\\=\\<\end{matrix} \frac{a}{x},$$

or as $$\frac{n+1}{r} \begin{matrix}>\\=\\<\end{matrix} 1+\frac{a}{x},$$

or as $$\frac{n+1}{1+\frac{a}{x}} \begin{matrix}>\\=\\<\end{matrix} r.$$

Since n is a fraction, $\left(\dfrac{n+1}{r}-1\right)$ can never be made $=0$; and by taking r great enough (that is, by taking a sufficient number of terms), $\left(\dfrac{n+1}{r}-1\right)$ can be made as near (-1) as we please, and therefore $\left(\dfrac{n+1}{r}-1\right)\dfrac{x}{a}$ can be made as near $-\dfrac{x}{a}$ as we please. In this case whether there will be a greatest term or not depends upon the value of $\dfrac{x}{a}$. If $\dfrac{x}{a} \overset{>}{=} 1$ after a sufficient number of terms have been taken, each term will be equal to or greater than the preceding, and there will be no greatest term; but if $\dfrac{x}{a}<1$, there will be a greatest term, and it will be found from the condition that $\dfrac{n+1}{1+\dfrac{a}{x}}>r$. As in Art. 264, the greatest term will be the $(r+1)^{\text{th}}$ term when $r=\dfrac{n+1}{1+\dfrac{a}{x}}$, if r is an integer. If $\dfrac{n+1}{1+\dfrac{a}{x}}$ is not an integer, the greatest term will be the $(m+1)^{\text{th}}$, where m equals the integral part of $\dfrac{n+1}{1+\dfrac{a}{x}}$.

III. Let n be negative.

If n is negative let it $=-m$ (where m is positive).

Then $$\left(\frac{n-r+1}{r}\right)\cdot\frac{x}{a}=\left(\frac{-m-r+1}{r}\right)\frac{x}{a}=-\left(\frac{m+r-1}{r}\right)\cdot\frac{x}{a}.$$

As what is required is the *numerically* greatest term of the expansion, the negative sign may be neglected, and the multiplier will be $\left(\dfrac{m+r-1}{r}\right)\dfrac{x}{a}$. Then the $(r+1)^{\text{th}}$ term $\overset{>}{\underset{<}{=}}$ r^{th} term

according as $$\left(\frac{m+r-1}{r}\right)\frac{x}{a} \gtreqless 1,$$

or as $$\left(\frac{m-1}{r}+1\right)\frac{x}{a} \gtreqless 1.$$

As in the preceding case, if m is fractional, $\left(\frac{m-1}{r}+1\right)$ may be made as near unity as we please, and $\therefore \left(\frac{m-1}{r}+1\right)\frac{x}{a}$ as near $\frac{x}{a}$ as we please. If $\frac{x}{a} \geqq 1$, there will be no greatest term; but if $\frac{x}{a} < 1$, the greatest term will be found from the condition that

$$\left(\frac{m-1}{r}+1\right)\frac{x}{a} > 1,$$

or $$\frac{m-1}{r} > \frac{a}{x} - 1,$$

or $$\frac{m-1}{\frac{a}{x}-1} > r.$$

The rest of the proof is the same as in I. and II. when $\frac{m-1}{r}$ is positive.

If, however, $\frac{m-1}{r}$ is negative, a new case arises, for then $\left(\frac{m-1}{r}+1\right)$ is always < 1, and $\left(\frac{m-1}{r}+1\right)\frac{x}{a}$ is always < 1 if $x \leqq a$. Therefore the successive terms of the expansion will each be less than the preceding, and therefore the first term will be the greatest. If $x > a$ the greatest term will be found as in I. and II.

Ex. 1. – Find the greatest term in the expansion of $(a+x)^n$ when $n=\frac{19}{2}$ and $4x=3a$.

Here the $(r+1)^{\text{th}}$ term is > the r^{th} as long as

$$\left\{\frac{\frac{19}{2}-r+1}{r}\right\}\frac{3}{4}>1,$$

or

$$\frac{\frac{21}{2}-r}{r}>\frac{4}{3},$$

or

$$\frac{21}{2r}>\frac{7}{3}, \text{ or } \frac{9}{2}>r.$$

Therefore the greatest term is the fifth.

Ex. 2.—Find the greatest term in the expansion of $(1-\frac{2}{3}x)^{-n}$ when $n=2$ and $x=\frac{3}{4}$.

The $(r+1)^{\text{th}}$ term $\geqq$ r^{th} as long as

$$\left(\frac{2+r-1}{r}\right)\frac{1}{2}\geqq 1, \left\{\text{since } \frac{2x}{3}=\frac{1}{2}\right\}$$

or

$$\left(1+\frac{1}{r}\right)\geqq 2, \text{ or } 1\geqq r.$$

Therefore first and second terms are the greatest.

EXERCISE XXXI.

Find the greatest term in the expansion of:

1. $(1-x)^{-\frac{3}{2}}$ when $x=\frac{8}{9}$. 2. $(1-x)^{-\frac{17}{2}}$ when $x=\frac{3}{4}$.

3. $(1+x)^{-12}$ when $x=\frac{7}{9}$. 4. $(1-x)^{\frac{12 2}{5}}$ when $x=\frac{5}{6}$.

5. $(1+x)^{\frac{21}{2}}$ when $x=\frac{2}{3}$. 6. $(2x+5y)^{12}$ when $x=8$, $y=3$.

7. $(3x^2+5y^3)^{-n}$ when $x=9$, $y=2$, $n=15$.

Find the first negative term in the expansion of:

8. $(1+\frac{4}{3}x)^{\frac{27}{4}}$. 9. $(1+\frac{2}{3}x)^{\frac{21}{4}}$.

10. If a_1, a_2, a_3, a_4 be any four consecutive terms of an expanded binomial, prove that

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}.$$

SPECIAL APPLICATIONS OF THE BINOMIAL THEOREM.

1. Find the sum of the coefficients of the first $(r+1)$ terms of $(1-x)^n$.

Let a_0, a_1, a_2, $a_3 \ldots . a_r \ldots .$ be the coefficients of $(1-x)^n$.

Then $$(1-x)^n=a_0+a_1x+a_2x^2+a_3x^3+\ldots . a_rx^r+\ldots . ; \qquad (1)$$

also, $$(1-x)^{-1}=1+x+x^2+x^3+\ldots . x^r+\ldots . \qquad (2)$$

By inspection it is seen that the coefficient of x^r in the product of (1) and (2) $=a_0+a_1+a_2+\ldots . a_r$. But the coefficient of x^r in the product of (1) and (2) = coefficient of x^r in $(1-x)^n(1-x)^{-1}$ or $(1-x)^{n-1}$, and the coefficient of x^r in

$$(1-x)^{n-1}=\frac{(n-1)(n-2)\ldots . (n-r)}{\underline{|r}}(-1)^r.$$

$$\therefore a_0+a_1+a_2+\ldots . a_r=\frac{(n-1)(n-2)\ldots . (n-r)}{\underline{|r}}(-1)^r.$$

Ex.—Find the sum of the first $(r+1)$ coefficients of $(1-x)^{-3}$.

Let $\qquad (1-x)^{-3}=a_0+a_1x+a_2x^2+\ldots\,a_rx^r+\ldots,$

also, $\qquad (1-x)^{-1}=1+x+x^2+\ldots$

$\therefore\ a_0+a_1+a_2+\ldots\,a_r=$ coefficient of x^r in $(1-x)^{-3}(1-x)^{-1}$ or

$$(1-x)^{-4}=\frac{4\,.\,5\,.\,6\ldots(r+3)}{\lfloor r}=\frac{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\ldots(r+3)}{1\,.\,2\,.\,3\ \ldots\ r}$$

$$=\frac{(r+1)(r+2)(r+3)}{6}.$$

2. Find to five places of decimals the value of $\sqrt{98}$.

$$\sqrt{98}=\sqrt{100-2}=\sqrt{100\left(1-\frac{1}{50}\right)}=10\sqrt{1-\frac{1}{50}}=10\left(1-\frac{1}{50}\right)^{\frac{1}{2}};$$

but $10\left(1-\frac{1}{50}\right)^{\frac{1}{2}}=10\Bigg\{1-\left(\frac{1}{2}\right)\cdot\frac{1}{50}$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{\lfloor 2}\cdot\left(\frac{1}{50}\right)^2-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{\lfloor 3}\cdot\left(\frac{1}{50}\right)^3+\ldots\Bigg\}$$

$$=10\left\{1-\frac{1}{100}-\frac{1}{8\times 2500}-\frac{1}{16\times(50)^3}-\ldots\right\}$$

$$=10\left\{1-\frac{1}{100}-\frac{1}{20000}-\frac{1}{2000000}-\ldots\right\}$$

$$=10-\frac{1}{10}-\frac{1}{2000}-\frac{1}{200000}.$$

To obtain the values of the several fractions as decimals we proceed as follows:

$$\frac{1}{10}=\cdot 1,$$

$$\frac{1}{2000}=\frac{1}{2}\times\frac{1}{1000}=\frac{1}{2}(\cdot 001)=\cdot 0005,$$

$$\frac{1}{200000} = \frac{1}{100}\left(\frac{1}{2000}\right) = \cdot 000005.$$

$$\therefore\ 10 - \frac{1}{10} - \frac{1}{2000} - \frac{1}{200000} = 10 - (\cdot 100505) = 9\cdot 899495.$$

Similar artifices can be applied to find the values correct to a given number of decimal places of such expressions as $\sqrt[3]{126}$, $\sqrt[4]{2400}$, etc.

3. Show that the coefficient of x^{3n} in the expansion of $\dfrac{(1+x^2)^3}{(1-x^3)^2}$ is $2n$.

$$(1+x^2)^3(1-x^3)^{-2}$$
$$= (1+3x^2+3x^4+x^6)\{1+2x^3+3x^6+4x^9+\ldots\ldots(r+1)(x^3)^r+\ldots\ldots\}.$$

The coefficient required consists of all the terms in the product of these two expressions containing x^{3n}; and since every term of the expansion of $(1-x^3)^{-2}$ contains powers of x^3, and only one term of $(1+x^2)^3$, viz., x^6, contains a power of x^3, therefore the coefficient required = (coefficient of x^{3n} + coefficient of x^{3n-6}) in $(1-x^3)^{-2}$. The coefficient of x^{3n} in $(1-x^3)^{-2} = n+1$, and coefficient of x^{3n-6} or $x^{(n-2)3} = n-1$. Therefore coefficient of x^{3n} in product of $(1+x^2)^3(1-x^3)^{-2} = n+1+n-1 = 2n$.

4. The following are illustrations of approximations:

Ex. 1.—If x be so small that its square and higher powers may be neglected, find the value of

$$(1-7x)^{\frac{1}{3}}(1+2x)^{-\frac{3}{4}}.$$

Expanding and neglecting powers of x higher than x^2,

$$(1-7x)^{\frac{1}{3}}(1+2x)^{-\frac{3}{4}} = (1 - \frac{7}{3}x - \frac{49}{9}x^2 + \ldots\ldots)(1 - \frac{3}{2}x + \frac{21}{8}x^2 - \ldots\ldots)$$

$$= 1 - \frac{23}{6}x + \frac{49}{72}x^2 - .\quad . = 1 - \frac{23}{6}x, \text{ nearly.}$$

Ex. 2.—If $p-q$ be small compared with p or q, prove

$$\sqrt[n]{\frac{p}{q}}=\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}.$$

Assume $\sqrt[n]{\frac{p}{q}}=1+x$, where x is very small,

$$\therefore\ \frac{p}{q}=(1+x)^n=1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\dots.$$

For the first approximation neglect terms containing powers higher than x.

Then $$\frac{p}{q}=1+nx;\quad \therefore\ x=\frac{p-q}{nq}.$$

Again, for a second approximation retain term containing x^2.

Then $$\frac{p}{q}=1+nx+\frac{n(n-1)}{2}x^2=1+nx(1+\frac{n-1}{2}x)$$

$$=1+nx\left(1+\frac{n-1}{2}\cdot\frac{p-q}{nq}\right).$$

Simplifying, $$x=\frac{2(p-q)}{nq+np-p+q}.$$

$$\therefore\ 1+x=\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}=\sqrt[n]{\frac{p}{q}}.$$

5. Show that the integral part of $(5+2\sqrt{6})^n$ is odd if n be a positive integer.

Since $$(5+2\sqrt{6})\times(5-2\sqrt{6})=25-24=1,$$

$(5-2\sqrt{6})$ must be less than 1, $(5+2\sqrt{6})$ being greater than 1.

$$\therefore\ (5-2\sqrt{6})^n=\text{a proper fraction}=f_1.$$

Let $(5+2\sqrt{6})^n=I+f$, when I stands for the integral part and f the fractional part of the expansion.

If $(5+2\sqrt{6})^n$ and $(5-2\sqrt{6})^n$ are expanded, the *odd* terms will be identical in the two series, while the *even* terms will differ only in sign.

$$\therefore\ (5+2\sqrt{6})^n+(5-2\sqrt{6})^n=\begin{Bmatrix}\text{twice the sum of odd}\\ \text{terms of } (5+2\sqrt{6})^n\end{Bmatrix}$$
$$=\text{an even integer;}$$

$$\therefore\ I+f+f_1=\text{an even number}=N;$$

$$\therefore\ f+f_1=N-I=\text{an integer.}$$

But f and f_1 are each <1,

$$\therefore\ f+f_1=1=N-I;$$

$$\therefore\ I=N-1=\text{an odd integer, since } N \text{ is even.}$$

6. The Binomial Theorem is sometimes used to expand an expression of more than *two* terms.

Ex. 1.—Expand $(1+x+x^2)^4$.

$$\begin{aligned}(1+x+x^2)^4&=\{1+x(1+x)\}^4\\&=1+4x(1+x)+6x^2(1+x)^2+4x^3(1+x)^3+x^4(1+x)^4\\&=1+4x+4x^2+6x^2(1+2x+x^2)+4x^3(1+3x+3x^2+x^3)\\&\qquad+x^4(1+4x+6x^2+4x^3+x^4)\\&=1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8.\end{aligned}$$

Ex. 2.—Expand $(1+2x+3x^2+4x^3+\ldots.\infty)^3$.

$$\begin{aligned}(1+2x+3x^2+\ldots.\infty)^3&=\{(1-x)^{-2}\}^3=(1-x)^{-6}\\&=1+6x+21x^2+56x^3+\ldots.\end{aligned}$$

7. The following example is well worth noting:

Ex.—Find the sum of the first $n+r$ coefficients of $\dfrac{(1+x)^n}{(1-x)^2}$.

$$\begin{aligned}\frac{(1+x)^n}{(1-x)^2}&=(1+x)^n(1-x)^{-2}\\&=\left(1+nx+\frac{n(n-1)}{\underline{|2}}x^2+\ldots.\right)(1+2x+3x^2+\ldots.)\\&=a_0+a_1x+a_2x^2+\ldots.\ a_{n+r}x^{n+r}+\ldots.\end{aligned}$$

$$\therefore \frac{(1+x)^n}{(1-x)^3} = (a_0 + a_1 x + a_2 x^2 + \ldots .)(1-x)^{-1}$$
$$= (a_0 + a_1 x + a_2 x^2 + \ldots .)(1 + x + x^2 + \ldots .).$$

If from the product of

$$(a_0 + a_1 x + a_2 x^2 + \ldots .)(1 + x + x^2 + \ldots .)$$

the coefficient of x^{n+r-1} is selected, it will be found to be

$$a_0 + a_1 + a_2 + \ldots . + \ldots . a_{n+r-1},$$

or the sum required. Therefore the sum required must be equal to the coefficients of x^{n+r-1} in

$$\frac{(1+x)^n}{(1-x)^3}; \text{ that is, in } \{2-(1-x)\}^n(1-x)^{-3}.$$

Now, $\{2-(1-x)\}^n(1-x)^{-3} = 2^n(1-x)^{-3} - n \,.\, 2^{n-1}(1-x)^{-2}$

$$+ \frac{n(n-1)}{\lfloor 2} \,.\, 2^{n-2}(1-x)^{-1} - \frac{n(n-1)(n-2)}{\lfloor 3} 2^{n-3} + \ldots .$$

terms containing powers of $(1-x)$, with positive indices, of which the highest is $(1-x)^{n-3}$.

We must therefore select the coefficient of x^{n+r-1} from the terms containing $(1-x)$ with a negative index, that is, from

$$2^n(1-x)^{-3}, \quad -n \,.\, 2^{n-1}(1-x)^{-2}, \quad \frac{n(n-1)}{2} 2^{n-2}(1-x)^{-1}.$$

The coefficients of x^{n+r-1} in $(1-x)^{-3}$, $(1-x)^{-2}$ and $(1-x)^{-1}$ are respectively

$$\frac{(n+r)(n+r+1)}{2}, \quad n+r \text{ and } 1.$$

Therefore the whole coefficient

$$= 2^{n-1}(n+r)(n+r+1) - 2^{n-1}n(n+r) + 2^{n-3}n(n-1).$$

8. Series are often summed by observing that they are composed of one or more binomial expansions.

Ex. 1.—Find the sum to infinity of

$$2^n\{1 - n \cdot \frac{1-x}{1+x} + \frac{n(n+1)}{\lfloor 2}\left(\frac{1-x}{1+x}\right)^2 - \frac{n(n+1)(n+2)}{\lfloor 3}\left(\frac{1-x}{1+x}\right)^3 + \ldots\}.$$

By inspection it is seen that

$$1 - \frac{n(1-x)}{1+x} + \frac{n(n+1)}{\lfloor 2}\left(\frac{1-x}{1+x}\right)^2 - \ldots$$

$$= \left(1 + \frac{1-x}{1+x}\right)^{-n} = \left(\frac{2}{1+x}\right)^{-n} = \frac{(1+x)^n}{2^n}.$$

$$\therefore\ 2^n\{1 - \frac{n(1-x)}{1+x} + \ldots\} = (1+x)^n.$$

Ex. 2.—Sum to infinity, $1 + \frac{2}{6} + \frac{2.5}{6.12} + \frac{2.5.8}{6.12.18} + \ldots$.

Arrange the series as follows:

$$1 + \frac{\frac{2}{3}}{\lfloor 1} \cdot \left(\frac{1}{2}\right) + \frac{\frac{2}{3} \cdot \frac{5}{3}}{\lfloor 2} \cdot \left(\frac{1}{2}\right)^2 + \frac{\frac{2}{3} \cdot \frac{5}{3} \cdot \frac{8}{3}}{\lfloor 3} \cdot \left(\frac{1}{2}\right)^3 + \ldots.$$

By inspection this is seen to be the expansion of

$$\left(1 - \frac{1}{2}\right)^{-\frac{2}{3}} = \left(\frac{1}{2}\right)^{-\frac{2}{3}} = 2^{\frac{2}{3}} = \sqrt[3]{4}.$$

9. The sum of a series is frequently found by observing that it is the coefficient of some power of x (or other quantity) in a series formed by multiplying together, or otherwise combining, two or more series.

Ex.—Show that if n be a positive integer not less than 4,

$$1 - 4n + \frac{4.5}{\lfloor 2} \cdot \frac{n(n-1)}{\lfloor 2} - \frac{4.5.6}{\lfloor 3} \cdot \frac{n(n-1)(n-2)}{\lfloor 3} + \ldots = 0.$$

$$(1+x)^{-4}=1-4x+\frac{4.5}{\lfloor 2}.x^2-\frac{4.5.6}{\lfloor 3}.x^3+\dots. \qquad (1)$$

and

$$\left(1+\frac{1}{x}\right)^n=1+n.\frac{1}{x}+\frac{n(n-1)}{\lfloor 2}.\frac{1}{x^2}+\frac{n(n-1)(n-2)}{\lfloor 3}.\frac{1}{x^3}+\dots. \qquad (2)$$

It is evident that

$$1-4n+\frac{4.5}{\lfloor 2}.\frac{n(n-1)}{\lfloor 2}-\dots.$$

= the coefficient of x^0 in product of (1) and (2) = the coefficient of x^0 in

$$(1+x)^{-4}\left(1+\frac{1}{x}\right)^n \text{ or } \frac{(1+x)^{n-4}}{x^n}.$$

But every term of $\dfrac{(1+x)^{n-4}}{x^n}$ contains x (with a negative index);

∴ the series = 0.

276. *To find the number of homogeneous products of r dimensions that can be obtained from n letters.*

Let $a, b, c, d\dots.$ be the n letters. Take n series,

$$\begin{array}{l} 1+ax+a^2x^2+a^3x^3+\dots.\infty, \\ 1+bx+b^2x^2+b^3x^3+\dots.\infty, \\ 1+cx+c^2x^2+c^3x^3+\dots.\infty, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{array}$$

in which $ax, bx, cx\dots.$ are each <1.

It is evident that if these n series be multiplied together, the coefficient of x will be the products of *one* dimension obtained from $a, b, c, d\dots.$, the coefficient of x^2 will be the products of *two* dimensions, and, generally, the coefficients of x^r will be the products of r dimensions from the given letters and their powers.

The problem, then, is to find the number of terms forming the coefficient of x^r in the product of

$$(1+ax+a^2x^2+\ldots\ldots)(1+bx+b^2x^2+\ldots\ldots)(1+cx+c^2x^2+\ldots\ldots)(\ldots\ldots)\ldots\ldots$$

Now,
$$\begin{aligned}
1+ax+a^2x^2+a^3x^3+\ldots\ldots\infty &= (1-ax)^{-1} \text{ when } ax<1;\\
1+bx+b^2x^2+b^3x^3+\ldots\ldots\infty &= (1-bx)^{-1} \text{ when } bx<1;\\
1+cx+c^2x^2+c^3x^3+\ldots\ldots\infty &= (1-cx)^{-1} \text{ when } cx<1;\\
\ldots\ldots\ldots\ldots &= \ldots\ldots\ldots\ldots
\end{aligned}$$

$$\begin{aligned}
\therefore\ (1+ax+a^2x^2\ldots\ldots)(1+bx+b^2x^2+\ldots\ldots)(1-cx+c^2x^2\ldots\ldots)\ldots\ldots\\
=(1-ax)^{-1}(1-bx)^{-1}(1-cx)^{-1}\ldots\ldots
\end{aligned}$$

The number of terms forming the coefficient of x^r, therefore, will be the number of terms in the coefficient of x^r in

$$(1-ax)^{-1}(1-bx)^{-1}(1-cx)^{-1}\ldots\ldots$$

But the number of terms in the coefficient will not be altered by giving a, b, c, $d\ldots\ldots$ the particular value 1, in which case each term will be 1, and the coefficient will be the number of terms. If $a=b=c=d\ldots\ldots=1$,

$$(1-ax)^{-1}(1-bx)^{-1}(1-bx)^{-1}\ldots\ldots=(1-x)^{-n},$$

and the coefficient of x^r in

$$\begin{aligned}
(1-x)^{-n} &= \frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{\underline{|r}}\\
&= \frac{\underline{|n+r-1}}{\underline{|n-1}\ \underline{|r}}.
\end{aligned}$$

Therefore the sum of the homogeneous products of r dimensions of a, b, c, $d\ldots\ldots$

$$=\frac{\underline{|n+r-1}}{\underline{|r}\ \underline{|n-1}}.$$

277. *To find the number of terms in the expansion of any multinomial when the index is a positive integer.*

Let $(a_1+a_2+a_3+\ldots . a_m)^n$ be the multinomial to be expanded. Every term of the expansion will be of n dimensions, and therefore the number of terms in the expansion will be the number of terms of n dimensions formed from m letters and their powers, which by the preceding Art.

$$=\frac{\lfloor m+n-1}{\lfloor n \, \lfloor m-1}.$$

278. From the result of Art. 277 we can readily find the number of combinations of n things, r at a time, when repetitions are allowed.

Let the n things be the n letters, $a, b, c, d \ldots .$; then the number of combinations of n things, r at a time, when repetitions are allowed, will be the number of homogeneous products of r dimensions formed from the n letters, $a, b, c, d \ldots .$, and their powers, and therefore will be

$$=\frac{\lfloor n+r-1}{\lfloor r \, \lfloor n-1}.$$

This result was proved in the chapter on Combinations, Art. 239.

279. Very frequently the method of Art. 276 is found useful in the solution of difficult problems in Permutations and Combinations.

Ex.—In how many ways can 20 be thrown with 4 dice, each of which has six faces marked 1, 2, 3, 4, 5, 6 respectively?

Let the n faces be marked as follows: $a, a^2, a^3 \ldots . a^6$; then the coefficient of x^{20} in $(1+ax+a^2x^2+a^3x^3+\ldots . a^6x^6)^4$ will be the different ways 20 can be thrown with the four dice; and the *number* of ways will be found by putting $a=1$, for then each term of the

coefficient will become 1. If $a = 1$, we have to find the coefficient of x^{20} in $(1 + x + x^2 + x^3 + \ldots . x^6)^4$, or the coefficient of x^{20} in

$$\left(\frac{1 - x^7}{1 - x}\right)^4 \text{ or in } (1 - x^7)^4(1 - x)^{-4}.$$

Now, $(1 - x^7)^4(1 - x)^{-4}$

$$= (1 - 4x^7 + 6x^{14} - \ldots .)\left\{1 + 4x + 6x^2 + \ldots . \frac{(r+1)(r+2)(r+3)}{\underline{|3}} x^r \ldots .\right\}.$$

Selecting from the product of these two expansions the terms containing x^{20} we obtain

$$\frac{21 \,.\, 22 \,.\, 23}{\underline{|3}} - \frac{4(14 \,.\, 15 \,.\, 16)}{\underline{|3}} + \frac{6(7 \,.\, 8 \,.\, 9)}{\underline{|3}} = 35.$$

EXERCISE XXXII.

1. Find the coefficient of x^{20} in the expansion of $\dfrac{1 - 2x}{(1 - x)^2}$.

2. Find the coefficient of x^m in the expansion of $\dfrac{3x^2 - 2}{x + x^2}$.

3. Find the coefficient of x^m in the expansion of $\dfrac{2 + x + x^2}{(1 - x)^3}$.

4. Show that the coefficient of x^{n+r-1} in the expansion of

$$\frac{(1 + x)^n}{(1 - x)^2} \text{ is } 2^{n-1}(n + 2r).$$

5. Show that the coefficient of x^{n+r-1} in the expansion of

$$\frac{(1 - 3x)^n}{(1 - 2x)^2} \text{ is } (-1)^n(r - 2n)2^{r-1}.$$

6. Find to four places of decimals the value of:

(1) $\sqrt[3]{999}$; (2) $\sqrt[3]{1002}$; (3) $\sqrt[4]{2400}$; (4) $\sqrt[5]{3129}$.

7. If x be very small, show that

$$\frac{(1 - 3x)^{-\frac{2}{3}} + (1 - 4x)^{-\frac{3}{4}}}{(1 - 3x)^{-\frac{1}{3}} + (1 - 4x)^{-\frac{1}{4}}} = 1 + \frac{3x}{2}.$$

8. If x be very small, find the value of

$$(1)\ \frac{(8+3x)^{\frac{2}{3}}}{(2+3x)\sqrt{4-5x}};\quad (2)\ \sqrt{4-x}\left(3-\frac{x}{2}\right)^{-1}.$$

9. If N and n be nearly equal, then $\sqrt{\dfrac{N}{n}}=\dfrac{N}{N+n}+\dfrac{1}{4}\cdot\dfrac{N+n}{N}$, very nearly. If $\dfrac{N}{N+n}$ and $\dfrac{1}{4}\cdot\dfrac{N+n}{n}$ have their first p decimal places the same, show that the approximation may be relied on to $2p$ decimal places.

10. If $c=a-b$, and it be very small compared with a and b, then $a^2b^2(a^2-a^2x^2+b^2x^2)^{-\frac{3}{2}}=a-2c+3cx^2$, nearly.

11. If $(6\sqrt{6}+14)^{2n+1}=N$, and f be its fractional part, then $Nf=20^{2n+1}$.

12. Show that the integral part of $(8+3\sqrt{7})^n$ is odd if n be a positive integer.

13. If $(3\sqrt{3}+5)^{2r+1}=I+f$, where I is an integer and f a proper fraction, then will $f(I+f)=2^{2r+1}$.

14. If n be a positive integer, the integer next greater than $(3+\sqrt{5})^n$ is divisible by 2^n.

15. Find the coefficient of x^4 in $(1-2x-2x^2)^{\frac{7}{2}}$.

16. Find the coefficient of x^r in $(4a^2+6ax+9x^2)^{-1}$.

17. Show that the coefficient of x^{3m} in $\dfrac{1+x}{(1+x+x^2)^2}$ is $2m+1$.

18. Find the coefficient of x^r in the expansion of

$$(1\,.\,2+2\,.\,3x+3\,.\,4x^2+\ldots\ldots\infty)^2.$$

19. Show that the coefficient of x^r in the expansion of

$$(1+x+2x^2+3x^3+\ldots\ldots)^2 \text{ is } \frac{r(r^2+11)}{6}.$$

20. Show that the coefficient of x^n in the expansion of $\dfrac{1}{1+x+x^2}$ is 1, 0 or -1, according as n is of the form $3m$, $3m-1$ or $3m+1$.

21. Prove $\sqrt{8} = 1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \dots$

22. Prove $\sqrt{\frac{2}{3}} = 1 - \frac{1}{2} \cdot \frac{1}{2} + \frac{1.3}{2.4} \cdot \frac{1}{2^2} - \frac{1.3.5}{2.4.6} \cdot \frac{1}{2^3} + \dots$

23. Sum the series, $\frac{3.5}{3.6} + \frac{3.5.7}{3.6.9} + \frac{3.5.7.9}{3.6.9.12} + \dots$

24. Prove $\frac{1}{4} + \frac{1.3}{4.6} + \frac{1.3.5}{4.6.8} + \dots \infty = 1.$

25. Prove $1 + \frac{3}{8} + \frac{3.5}{8.10} + \frac{3.5.7}{8.10.12} + \dots \infty = 2.$

26. Prove $1 + \frac{11}{14} + \frac{11.13}{14.16} + \frac{11.13.15}{14.16.18} + \dots \infty = 12.$

27. Prove $1 + \frac{2n}{3} + \frac{2n(2n+2)}{3.6} + \frac{2n(2n+2)(2n+4)}{3.6.9} + \dots$

$$= 2^n\left\{1 + \frac{n}{3} + \frac{n(n+1)}{3.6} + \frac{n(n+1)(n+2)}{3.6.9} + \dots\right\}.$$

28. Prove $7^n\left\{1 + \frac{n}{7} + \frac{n(n-1)}{7.14} + \frac{n(n-1)(n-2)}{7.14.21} + \dots\right\}$

$$= 4^n\left\{1 + \frac{n}{2} + \frac{n(n+1)}{2.4} + \frac{n(n+1)(n+2)}{2.4.6} + \dots\right\}.$$

29. Prove $\dfrac{1 + n \cdot \dfrac{a}{a-b} + \dfrac{n(n+1)}{\lfloor 2}\left(\dfrac{a}{a-b}\right)^2 + \dots}{1 + n \cdot \dfrac{b}{b-a} + \dfrac{n(n+1)}{\lfloor 2}\left(\dfrac{b}{b-a}\right)^2 + \dots} = (-1)^n \dfrac{a^n}{b^n}.$

30. Show that if $x > -\frac{1}{2}$,

$$\frac{x}{\sqrt{x+1}} = \frac{x}{1+x} + \frac{1}{2}\left(\frac{x}{1+x}\right)^2 + \frac{1.3}{2.4}\left(\frac{x}{1+x}\right)^3 + \frac{1.3.5}{2.4.6}\left(\frac{x}{1+x}\right)^4 + \dots$$

31. Show that

$$(1+x)^{2n} = (1+x)^n + n(1+x)^{n-1} \,.\, x + \frac{n(n+1)}{\underline{|2}} x^2(1+x)^{n-2}$$
$$+ \frac{n(n+1)(n+2)}{\underline{|3}} x^3(1+x)^{n-3} + \dots.$$

32. Show that if $a < b$,

$$a^2b^2 = (a+b)^4 \left\{ \frac{a^2}{b^2} - \frac{4}{1} \cdot \frac{a^3}{b^3} + \frac{4 \,.\, 5}{1 \,.\, 2} \cdot \frac{a^4}{b^4} - \frac{4 \,.\, 5 \,.\, 6}{1 \,.\, 2 \,.\, 3} \cdot \frac{a^5}{b^5} + \dots \right\}.$$

33. Show that

$$1 - \frac{n+x}{1+x} + \frac{(n+2x)(n-1)}{\underline{|2}\,(1+x)^2} - \frac{(n+3x)(n-1)(n-2)}{\underline{|3}\,(1+x)^3} + \dots = 0.$$

34. Show that if the numerical value of y be less than one-third of that of x,

$$1 + n\left(\frac{2y}{x+y}\right) + \frac{n(n+1)}{\underline{|2}}\left(\frac{2y}{x+y}\right)^2 + \frac{n(n+1)(n+2)}{\underline{|3}}\left(\frac{2y}{x+y}\right)^3 + \dots$$
$$= 1 + n\left(\frac{2y}{x-y}\right) + \frac{n(n-1)}{\underline{|2}}\left(\frac{2y}{x-y}\right)^2 + \dots.$$

35. Prove $(n^2-2)^{\frac{1}{2}}$

$$= \frac{n^2-1}{n}\left\{1 - \frac{1}{2} \cdot \frac{1}{(n^2-1)^2} - \frac{1}{8} \cdot \frac{1}{(n^2-1)^4} - \frac{1}{16} \cdot \frac{1}{(n^2-1)^6} - \dots\right\}.$$

36. Prove that if n be an even integer,

$$\frac{1}{\underline{|1}\,\underline{|n-1}} + \frac{1}{\underline{|3}\,\underline{|n-3}} + \frac{1}{\underline{|5}\,\underline{|n-5}} + \dots + \frac{1}{\underline{|n-1}\,\underline{|1}} = \frac{2^{n-1}}{\underline{|n}}.$$

37. Prove $\dfrac{1.3.5\dots(2r-1)}{\underline{|r}} + \dfrac{1.3.5\dots(2r-3)}{\underline{|r-1}} \cdot \dfrac{3}{1}$

$$+ \frac{1.3.5\dots(2r-5)}{\underline{|r-2}} \cdot \frac{3.5}{\underline{|2}} + \dots = 2^r(1+r).$$

38. If $p_r=\dfrac{1.3.5\ldots.(2r-1)}{2.4.6\ldots.2r}$, prove

$$p_{2n+1}+p_1p_{2n}+p_2p_{2n-1}+\ldots.p_np_{n+1}=\frac{1}{2}.$$

39. If $p_r=\dfrac{1.3.5\ldots.(2r-1)}{2^r\lfloor r}$, prove

$$2\{p_{2n}-p_1p_{2n-1}+p_2p_{2n-2}+\ldots.(-1)^{n-1}p_{n-1}p_{n+1}\}=p_n+(-1)^{n-1}p_n^2.$$

40. If $a_0,\ a_1,\ a_2\ldots.a_n$ are the coefficients in the expansion of $(1+x)^n$ when n is a positive integer, prove

(1) $a_0-a_1+a_2-a_3+\ldots.(-1)^ra_r=(-1)^r\dfrac{\lfloor n-1}{\lfloor r\ \lfloor n-r-1}$.

(2) $a_0-2a_1+3a_2-4a_3+\ldots.(-1)^n(n+1)a_n=0$;

(3) $a_0^2-a_1^2+a_2^2-a_3^2+\ldots.(-1)^na_n^2=0$ or $(-1)^{\frac{n}{2}}a_{\frac{n}{2}}$.

41. Prove $1+3n+\dfrac{3.4}{1.2}\cdot\dfrac{n(n-1)}{\lfloor 2}+\dfrac{3.4.5}{1.2.3}\cdot\dfrac{n(n-1)(n-2)}{\lfloor 3}+\ldots$

$$=2^{n-3}(n^2+7n+8).$$

42. If $p_r=\dfrac{1.3.5\ldots.(2r-1)}{2.4.6\ldots.2r}$ and $q_r=\dfrac{5.7.9\ldots.(2r+3)}{2.4.6\ldots.2r}$, prove

$$p_r+p_{r-1}q_1+p_{r-2}q_2+\ldots.q_r=\frac{1}{2}(r+1)(r+2).$$

43. Prove

$$1+\frac{1}{6}\cdot\frac{n^2}{1}+\left(\frac{1}{6}\right)^2\left\{\frac{n(n-1)}{\lfloor 2}\right\}^2+\left(\frac{1}{6}\right)^3\left\{\frac{n(n-1)(n-2)}{\lfloor 3}\right\}^2+\ldots.$$

$$=\left(\frac{7}{6}\right)^n\left\{1+\frac{n(n-1)}{1}\cdot\left(\frac{6}{7^2}\right)+\frac{n(n-1)(n-2)(n-3)}{\{\lfloor 2\}^2}\cdot\left(\frac{6}{7^2}\right)^2+\ldots\right\}.$$

44. If a_r be the coefficient of x^r in $(1+x)^n$, prove that if k be less than n,

$$\frac{\lfloor n}{\lfloor n-k}-a_1\frac{\lfloor n-1}{\lfloor n-k-1}+a_2\frac{\lfloor n-2}{\lfloor n-k-2}-\ldots.=0.$$

45. If $x_n = x(x+1)(x+2)\ldots\ldots(x+n-1)$, show that

$$(x+y)_n = x_n + nx_{n-1}y_1 + \frac{n(n-1)}{\underline{|2}}x_{n-2}y_2 + \ldots\ldots y_n.$$

(This is Vandermonde's Theorem.)

46. Prove that if $(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots\ldots c_nx^n$, then

$$n(1+x)^{n-1} = c_1 + 2c_2x + 3c_3x^2 + \ldots\ldots nc_nx^{n-1}.$$

47. Prove, using the notation of the preceding example, that

$$c_1^2 + 2c_2^2 + 3c_3^2 + \ldots\ldots nc_n^2 = \frac{\underline{|2n-1}}{\underline{|n-1}\;\underline{|n-1}},$$

and $$c_0^2 + 2c_1^2 + 3c_2^2 + \ldots\ldots (n+1)c_n^2 = \frac{(n+2)\;\underline{|2n-1}}{\underline{|n}\;\underline{|n-1}}.$$

48. If $f(r)$

$$= \frac{\underline{|n}}{\underline{|r}\;\underline{|n-r}} + n\frac{\underline{|n}}{\underline{|r+1}\;\underline{|n-r-1}} + \frac{n(n-1)}{\underline{|2}}\cdot\frac{\underline{|n}}{\underline{|r+2}\;\underline{|n-r-2}} + .$$

then $f(0) + nf(1) + \dfrac{n(n-1)}{\underline{|2}}f(2) + \ldots f(n) = \dfrac{(2n+1)(2n+2)\ldots\ldots 3n}{\underline{|n}}.$

49. If $(1+x)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots$, then will

$\dfrac{a_1^2 + 2a_2^2 + 3a_3^2 + \ldots\ldots na_n^2}{a_0^2 + a_1^2 + a_2^2 + \ldots\ldots a_n^2} = \dfrac{n}{2}$, n being a positive integer.

50. Prove $2^n - (n-1)2^{n-2}$

$$+ \frac{(n-2)(n-3)}{\underline{|2}}2^{n-4} - \frac{(n-3)(n-4)(n-5)}{\underline{|3}}2^{n-6} + \ldots\ldots = n+1.$$

51. Prove

$$2^n + \frac{n(n-1)}{1^2}\cdot 2^{n-2} + \frac{n(n-1)(n-2)(n-3)}{1^2 \cdot 2^2}\cdot 2^{n-4} + \ldots\ldots = \frac{\underline{|2n}}{\underline{|n}\;\underline{|n}}.$$

52. Prove that the sum of the first n coefficients of the expansion, in ascending power of x, of

$\dfrac{(1+x)^n}{(1-x)^3}$ is $\dfrac{n(n+2)(n+7)}{3}2^{n-1}$, n being a positive integer.

53. In a shooting competition a man can score 5, 4, 3, 2, 1 or 0 points for each shot. Find the number of different ways in which he can score 30 in 7 shots.

54. A man goes in for an examination in which there are four papers with a maximum of m marks for each paper. Show that the number of ways of getting half marks on the whole is

$$\frac{1}{3}(m+1)(2m^2+4m+3).$$

55. There are two regular polyhedrons marked in the manner of dice, and the numbers of their faces are m, $m+n$ respectively. How many different throws can possibly be made by throwing them together?

56. If, in the preceding question, the number of polyhedrons be four, and the numbers on their faces 3, 6, 8, 12 respectively, show that the number of different throws that can be made by throwing all together is 552.

57. If $s=a^2+b^2$, $p=2ab$, $P=(a+b)^p$, show that

$$P \,.\, P^{\frac{1}{2}} \,.\, P^{\frac{1}{4}} \,.\, P^{\frac{1}{8}} \ldots . \infty$$
$$= s^p + p^2 s^{p-1} + p^3 \,.\, \frac{p-1}{2} s^{p-2} + p^4 \,.\, \frac{p-1}{2} \,.\, \frac{p-2}{3} s^{p-3} + \ldots .$$

58. If $f(n, m) = \frac{1}{m} - n\left(\frac{1}{m+p}\right) + \frac{n(n-1)}{\underline{|2}}\left(\frac{1}{m+2p}\right) - \ldots .$, show that $f(n, m) = \left(\frac{np}{m}\right) f(\overline{n-1}, \overline{m+p})$.

59. If $z^2+z+1=0$, show that the sum of those terms of the expansion of $(1+x)^n$, in which the index of x is a multiple of 3,

$$= \frac{1}{3}\left\{(1+xz)^n + (1+x)^n + \left(1+\frac{x}{z}\right)^n\right\}.$$

60. If a_r = coefficient of x^r in $(1+x)^n$, show that

$$(a_0+a_1)(a_1+a_2)(a_2+a_3)+\ldots .(a_{n-1}+a_n) = \frac{(n+1)^n}{\underline{|n}} a_0, a_1, a_2 \ldots . a_n.$$

CHAPTER XXI.

INTEREST, DISCOUNT AND ANNUITIES.

1. Interest is money paid for the use of money. The sum for which interest is paid is called the **Principal**; the interest on one dollar for one year is called the **Rate**; and the sum of the principal and interest for any given time is the **Amount.**

2. Interest is of two kinds, simple and compound.

Simple Interest is interest reckoned upon the original sum only. But each payment of interest as it becomes due is usually added to the principal, and interest for the succeeding period, charged upon the whole. This is known as **Compound Interest.** The latter is the only correct method of reckoning interest which should evidently be paid upon the whole debt without regard to the manner in which the debt has been incurred.

3. *To find the interest and the amount of a given sum in a given time at simple interest.*

Let P be the given sum in dollars, r the interest on one dollar for one year, n the time in years, I the interest, and A the amount.

The interest of P for one year is Pr, and therefore for n years it is Pnr; that is

$$I = Pnr, \qquad (1)$$

And

$$A = P + I,$$

$$= P(1 + nr). \qquad (2)$$

From (1) and (2) it is evident that if any three of the quanti-

ties, P, I, n, r, A (excepting the three P, I, A) be given the other two may be found.

4. *To find the present value and discount of a given sum due in a given time, allowing simple interest.*

Let A be the given sum, r the given rate, n the number of years, P its present value, and D the discount required.

Then $$A = P\,(1 + nr)$$

Therefore $$P = \frac{A}{1 + nr} \qquad (3)$$

And $$D = A - P,$$

$$= A - \frac{A}{1 + nr}$$

$$= \frac{Anr}{1 + nr} \qquad (4)$$

In actual business, and for short periods of time, it is customary to deduct interest on the whole sum instead of the true discount which, as shown by (4), is the interest on the present worth. This is known as **Bank Discount,** which is therefore greater than true discount by the interest on the true discount.

5. *To find the amount and the interest of a given sum in a given time at compound interest.*

Let P denote the principal, r the rate, n the time in years, I the interest, and A the amount.

The amount at the end of a year is found by multiplying the principal at the beginning by $1 + r$, and the amount at the end of each year is the principal at the beginning of the next year, therefore the series.

$$P\,(1 + r),\ P\,(1 + r)^2,\ P\,(1 + r)^3 \ldots\ldots P\,(1 + r)^n$$

gives the amount at the end of $1, 2, 3 \ldots\ldots n$ years respectively.

Therefore $$A = P\,(1 + r)^n, \qquad (5)$$

And $$I = A - P$$

$$= P\,\{(1 + r)^n - 1\} \qquad (6)$$

6. *To find the present value and the discount on a given sum due in a given time, allowing compound interest.*

Let A denote the given sum, P its present value, r the rate, n the time in years, D the discount.

Then from (5) $$A = P(1+r)^n; \tag{7}$$

Therefore $$P = \frac{A}{(1+r)^n} \tag{8}$$

And $$D = A - P,$$

$$= A\left\{1 - \frac{1}{(1+r)^n}\right\} \tag{9}$$

7. In the examples which precede the time has been assumed to be an exact number of years. When a fraction of a year occurs in actual business it is customary to allow the same fraction of the annual rate of interest, but this is not theoretically correct, and frequently leads to contradictory results as the following simple example will show:

Ex.—A note for \$500 is drawn January 1st, due in one year with 6 per cent. interest; find its cash value on July 1st of the same year.

Two modes of solution present themselves, which appear equally reasonable. We may add 6 months' interest making \$515, or find the present worth of \$530, the sum due at the end of the year, 6 months before it is due, giving \$514.56. The discrepancy arises from our having twice assumed that 6 per cent. for a year is the same as 3 per cent. for 6 months, which assumption is untrue.

8. In actual business interest is sometimes payable more frequently than once a year, and then there is a difference between the *nominal* rate per annum and the *true annual rate*. Thus, 6 per cent. per annum payable half-yearly gives really $(1.03)^2 = 1.0609$, or $6\frac{9}{100}$ per cent. annually. If the interest is payable q times per annum, and if r is the nominal annual rate,

then the interest on one dollar for one interval is $\frac{r}{q}$, and since there are qn intervals in n years the amount of $\$P$ in n years will be $P\left(1+\frac{r}{q}\right)^{qn}$

In this case the interest is said to be capitalized q times a year.

9. In solving problems in compound interest logarithms are frequently useful to avoid tedious multiplications and are sometimes essential, as in the following example:

Ex.—In how many years will \$126 amount to \$672 at $4\frac{1}{6}$ per cent. compound interest; give log $2=30103$, log $3=.47712$. Let n be the number of years;

Then $$126\,(1.04\tfrac{1}{6})^n=672, \qquad \text{(Art. 5)}$$

Therefore $$\left(\frac{25}{24}\right)^n=\frac{672}{126}=\frac{16}{3},$$

or $$n\,(\log 25-\log 24)=\log 16-\log 3,$$

from which $$n=\frac{4\log 2-\log 3}{2-5\log 2-\log 3},$$
$$=41 \text{ nearly.}$$

EXERCISE XXXIII.

1. At simple interest the interest on a sum of money is \$135 and the discount \$120; find the sum of money and the rate per cent. for the given time.

2. The compound interest on a given sum for 5 years at 5 per cent. exceeds the simple interest for the same time by \$19.71; find the sum.

3. Show that the true discount of any sum is half the harmonic mean between the sum and its simple interest for the given time.

4. If the simple interest be $\frac{p}{q}$ of the principal, the true discount will be $\frac{p}{p+q}$ of the given sum.

5. The bank discount on a bill due in one year at 8 per cent. is \$540; find the true discount.

6. If the interest on \$$A$ for a year be equal to the discount on \$$B$ for the same time, find the rate of interest.

7. Divide \$1,000 between three persons aged 18, 19, and 20 years respectively, so that on their coming of age (21 years), their shares may be proportional to 4, 5, and 6, reckoning compound interest at 5 per cent.

8. In how many years will a given sum of money treble itself (1) at simple interest, (2) at compound interest, $3\frac{1}{2}$ per cent.; having given $\log 3 = .47712$, $\log 1035 = 3.01494$.

9. What sum of money at 6 per cent. compound interest will amount to \$1,000 in twelve years; given $\log 106 = 2.025306$, $\log 49697 = 4.696329$?

10. In what time will \$100 become \$1,050 at 5 per cent. compound interest; given $\log 2 = .301030$, $\log 3 = .477121$, $\log 7 = .845098$?

11. A merchant's profits during each year are $\frac{1}{m}$, and his expenses $\frac{1}{n}$ of his capital at the beginning of the year. In how many years will his capital be doubled?

12. A person invests his money in a business which pays 4 per cent. per annum. Each year he spends a sum equal to twice the original income. In how many years will he be ruined; given $\log 2 = .3010300$, $\log 13 = 1.1139434$.

13. Find the interest on *one cent* for 2,000 years at 5 per cent. (1) at simple interest (2) at compound interest; Given $\log 105 = 2.0211893$, $\log 23912 = 4.3786159$.

ANNUITIES.

10. An **Annuity** is a fixed sum of money payable at the end of equal intervals of time, usually one year each.

11. An annuity is said to be **forborne** when it is left unpaid for any number of years.

12. A **Deferred Annuity,** or **Reversion,** is an annuity which does not begin until the end of a certain number of years. When the annuity is deferred n years, it is said to begin after n years, but the payments being made at the *end* of each period, the first payment will be made at the end of $n+1$ years.

13. An annuity which is to continue for ever is called a **Perpetuity**.

14. A **Freehold Estate** consists of land, or other property, which yields a perpetual annuity usually called the **Rent.** The annual income derived from freehold estates, irredeemable stocks, etc., is frequently called a **Year's Purchase**.

15. Freehold estates are sometimes leased for a term of years for a certain sum in cash at the beginning of the period. Suppose an estate so leased, and that p years before the term expires the lessee wishes to obtain a new lease good for $p+n$ years, the sum which he must pay for this extension of time is called the **Fine** for renewing n years of the lease.

16. *To find the amount of an annuity left unpaid for a given number of years allowing* (1) *simple interest,* (2) *compound interest.*

Let P be the annual payment, r the interest on one dollar for one year, A the amount, and n the number of years.

Since the payments are made at the end of each year, the first payment will be at interest $n-1$ years, the second $n-2$ years, and so on. Thus, we have,

At simple interest from (2),

$$A = P\{1+(n-1)r\} + P\{1+(n-2)r\} + \dots . P(1+r) + P$$
$$= nP + (1+2+3+\dots . \overline{n-1})Pr$$
$$= nP + \frac{n(n-1)}{2}Pr. \qquad (10)$$

At compound interest from (5),

$$A = P(1+r)^{n-1} + P(1+r)^{n-2} + \dots . P(1+r) + P$$
$$= P\{1 + (1+r) + (1+r)^2 + \dots . (1+r)^{n-1}\}$$
$$= \frac{P}{r}\{(1+r)^n - 1\} \qquad (11)$$

The reader should observe that the *coefficients* $n-1$, $n-2$, etc., in the value of A at simple interest become *exponents* in the value at compound interest.

17. *To find the present value of an annuity to continue a given number of years, allowing* (1) *simple interest*, (2) *compound interest.*

With the notation of the preceding article, the values of the annuity at the *end* of the given time are,

$$nP + \frac{1}{2}n(n-1)Pr \text{ and } \frac{P}{r}\ \{(1+r)^n - 1\}$$

Therefore its values at the *beginning* of the given time will be found by dividing these expression by $1+nr$ and $(1+r)^n$ respectively (Arts. 4 and 6).

Thus,

$$\frac{nP + \frac{1}{2}n(n-1)Pr}{1+nr} \text{ and } \frac{P}{r}\left\{1 - \frac{1}{(1+r)^n}\right\}$$

are the required values by simple and compound interest respectively.

Cor. The value of a perpetuity may be found from the above

by making n infinitely great. The expression for the value at simple interest may be written $\dfrac{P+\frac{1}{2}(n-1)Pr}{\frac{1}{n}+r}$, and when $n=\infty$ the numerator becomes infinitely great, but the denominator remains finite, thus making the value of the present worth infinitely great. This is another indication of the impractical results of simple interest when long periods of time are involved.

In the expression for the value at compound interest, $\dfrac{1}{(1+r)^n}$ becomes indefinitely small when n becomes indefinitely great, and the present worth reduces to $\dfrac{P}{r}$, which is evidently correct, for this amount of cash will give a yearly interest of P for all time to come.

18. *To find the present value of a deferred annuity to commence at the end of* p *years, and to continue* n *years, allowing compound interest.*

The value at the beginning of the period of n years is

$$\frac{P}{r}\left\{1-\frac{1}{(1+r)^n}\right\} \qquad \text{(Art. 17)}$$

Therefore the value of this sum p years preceding this date is

$$\frac{P}{r(1+r)^p}\left\{1-\frac{1}{(1+r)^n}\right\} \qquad \text{(Art. 6)}$$

This formula also gives the fine to be paid for renewing n years of a lease, p years before the first lease expires.

Cor. The present value of a deferred perpetuity to commence after p years is $\dfrac{P}{r(1+r)^p}$.

Ex.—A mortgage for \$5000 with interest at 6 per cent. per annum is drawn January 1st, 1890, payable in 8 years; find its

cash value on March 1st, 1890, allowing the purchaser 10 per cent. per annum, payable half-yearly.

The annual payment of interest is \$300. The values of these payments at the end of the period of time with 10 per cent. interest, payable half-yearly, are 300 $(1.05)^{14}$, 300 $(1.05)^{12}$.... 300. Thus the total amount will be

$$\begin{aligned}
&5000+300\ \{(1.05)^{14}+(1.05)^{12}+\ldots .1\} \\
&=5000+300\left\{\frac{(1.05)^{16}-1}{(1.05)^{2}-1}\right\} \\
&=8462.06
\end{aligned}$$

We must now find the cash value of this sum payable in $7\frac{5}{8}$ years, allowing 10 per cent. interest, payable half-yearly. Thus,

$$\frac{8462.06}{(1.05)^{15\frac{1}{4}}}=\$3940.13 \qquad \text{(Art. 6)}$$

is the cash value required.

19. The preceding propositions are sufficient to solve the practical problems arising from loans, stocks, debentures, and all transactions involving periodical payments for a fixed number of years. Such are called **Annuities Certain,** but another and most extensive department of the subject relates to **Life Annuities,** which are payable during the life of a specified person or the survivor of a number of persons. For further information the student may consult the article "Annuities," in the "Encyclopædia Britannica," and "System and Tables of Life Insurance," by Levi W. Meech, Norwich, Conn.

EXERCISE XXXIV.

1. What yearly payment for three years will repay a loan of \$1,000, money being worth 6 per cent.?

2. Find the amount of an annuity of \$100 in 20 years, allowing compound interest at $4\frac{1}{2}$ per cent.; given

$$\log 1.045=.0191163, \ \log 24.117=1.3823260.$$

3. A freehold estate which rents for \$216 per annum is sold for \$4800; find the rate of interest.

4. How many years' purchase should be given for a freehold estate, money being worth $3\frac{1}{2}$ per cent.?

5. Find the cash value of an annuity of \$250 to continue for 20 years, money being worth 5 per cent.; given

$$(1.05)^{20} = 2.653297.$$

6. If a perpetual annuity is worth 25 years' purchase, find the amount of an annuity of \$625 at the end of 5 years.

7. What annuity, to continue 20 years, can be purchased for \$10,000, allowing compound interest at 5 per cent?

8. For what sum might an annuity of \$400 a year, for 10 years, to commence in 5 years, be purchased, allowing compound interest at 6 per cent.?

9. A person who enjoyed a perpetuity of \$1000 per annum provided in his will that, after his decease it should descend to his son for 10 years, to his daughter for the following 20 years, and to a benevolent society for ever after. What was the cash value of each bequest at the time of his decease, allowing compound interest at 6 per cent.? Given

$$(1.06)^{20} = 3.20713547.$$

10. If 25 years' purchase must be paid for an annuity to continue n years, and 30 years' purchase for an annuity to continue $2n$ years, find the rate per cent.

11. A man has a capital of \$20000, for which he receives interest at 5 per cent.; if he spends \$1800 every year, show that he will be ruined before the end of the seventeenth year; having given $\log 2 = .3010300$, $\log 3 = .4771213$, $\log 7 = .8450980$.

12. A young man enters upon a situation at a salary of \$100 per quarter, which is increased \$10 every payment. His expenses are \$75 for the first quarter, and increase 5 per cent. each succeeding quarter. He invests his savings at 6 per cent.

per annum. What will he be worth in 10 years? Proceeding in the same way would he ever be ruined?

13. A merchant invests \$12,000 which yields 25 per cent. profit annually. At the end of the first year he withdraws \$1000 for expenses, and each succeeding year $33\frac{1}{3}$ per cent. more than the preceding year. How many years before he will become bankrupt?

14. The annual rent of an estate is £500; if it is let on a lease of 20 years, calculate the fine to be paid to renew the lease when seven years have elapsed, allowing compound interest at 6 per cent.; having given

$$\log 106 = 2.0253059,\ \log 4.688385 = .6710233,\ \log 3.118042 = .4938820.$$

15. Find the present worth of a perpetual annuity of \$10 at the end of the first year, \$20 at the end of the second, and so on increasing \$10 each year; compound interest at 5 per cent. per annum.

16. An annuity is payable for a term of $2n$ years; show that its present worth for the first n years is $(1+r)^n$ times its present worth for the second n years. If the present worth for the whole time is m times the present worth for the last n years, find n.

17. A loan is repaid by an annual payment for n years of $\frac{\quad}{m}$ of the given sum; show that $(1+r)^n(1-mr)=1$.

18. If there be n annuities of $1, 2, 3 \ldots . n$ pounds respectively left unpaid for n years, find the sum of their amounts at simple interest.

19. If P be the present value of an annuity to continue for n years, and $P+Q$ its value for $2n$ years, find the yearly value of the annuity.

20. Find the present worth of $A, 2A, 3A \ldots . nA$ dollars due in $1, 2, 3 \ldots . n$ years respectively, allowing compound interest.

21. A freehold estate is to be held in succession by each of n charitable institutions for such times as will divide its value equally amongst them. For how many years does each hold it?

22. If a, b, c years' purchase must be paid for an annuity to continue n, $2n$, $3n$ years respectively; show that

$$a^2 - ab + b^2 = ac \text{ and } r = \frac{2b - a - c}{b^2 - ac}.$$

23. Show how each formula for simple interest may be derived from the corresponding formula for compound interest.

MISCELLANEOUS EXAMPLES.

EXERCISE XXXV.

1. Divide $1+x+x^2+x^3+x^4+x^6+x^7+x^8+x^9+x^{15}$ by $1-x^5+x^6$.

2. Show that $(x^2+xy+y^2)(a^2+ab+b^2)$

$$=(ax-by)^2+(ax-by)(ay+bx+by)+(ay+bx+by)^2.$$

3. If $s=a+b+c$, show that $(as+bc)(bs+ca)(cs+ab)$ is a complete square.

4. Find the G. C. M. of $nx^{n+1}-(n+1)x^n+1$ and $x^n-nx+n-1$.

5. Find the condition that the equations,

$$ax+by=1,$$
$$cx^2+dy^2=1,$$

may have but one solution, and find the solution.

6. If cx^2-ax+b divides ax^3-bx^2+c, then it will also divide bx^3-cx+a, and conversely.

7. Factor $(a+b)(a^3+b^3)^2+6ab(a+b)^2(a^3+b^3)+9a^2b^2(a+b)^3$.

8. Solve the equations,

$$(b+c)x+(c+a)y+(a+b)z=0,$$
$$(b-c)x+(c-a)y+(a-b)z=0,$$

$$\frac{x}{\frac{c}{a}-\frac{a}{b}}+\frac{y}{\frac{a}{b}-\frac{b}{c}}+\frac{z}{\frac{b}{c}-\frac{c}{a}}=n(ab+bc+ca).$$

9. Simplify the following, in which ω denotes a cube root of unity:

(1) $\dfrac{1}{x+y+z}+\dfrac{\omega}{x+\omega y+\omega^2 z}+\dfrac{\omega^2}{x+\omega^2 y+\omega z}$;

(2) $\dfrac{1}{x+y+z}+\dfrac{\omega^2}{x+\omega y+\omega^2 z}+\dfrac{\omega}{x+\omega^2 y+\omega z}$;

(3) $(x+y+z)^3+(x+\omega y+\omega^2 z)^3+(x+\omega^2 y+\omega z)^3-3(x^3+y^3+z^3-3xyz)$;

(4) $\dfrac{1}{(x+y+z)^3}+\dfrac{1}{(x+\omega y+\omega^2 z)^3}+\dfrac{1}{(x+\omega^2 y+\omega z)^3}-\dfrac{3}{x^3+y^3+z^3+3xyz}$.

10. A number consisting of three digits is doubled by reversing the digits. Find the digits in the scale of r, and hence show that there is no such number in the common scale. Find the scales in which it is possible, and show that the number formed by the first and last digits is also doubled by reversing its digits.

11. A parallelogram is inscribed in a triangle having two of its sides coinciding with sides of the triangle. Show, algebraically, that the parallelogram will have the greatest area when its sides are each half the corresponding sides of the triangle.

12. The sides of a triangle are the roots of $x^3-ax^2+bx-c=0$. Show that its area is $\dfrac{1}{2}\sqrt{a(4ab-a^3-8c)}$.

13. If

$$f(x)=a\left(\frac{1+\sqrt{5}}{2}\right)^x+b\left(\frac{1-\sqrt{5}}{2}\right)^x, \text{ then } f(x)+f(x+1)=f(x+2).$$

14. If $$2x=(18+5\sqrt{13})^n+(18-5\sqrt{13})^n$$

and $$2y\sqrt{13}=(18+5\sqrt{13})^n-(18-5\sqrt{13})^n,$$

show that $x^2-13y^2=(-1)^n$.

15. If $f_1(x)=\dfrac{1}{2}(e^x-e^{-x})$ and $f_2(x)=\dfrac{1}{2}(e^x+e^{-x})$,

then $$f_1(x)\cdot f_2(y)\pm f_2(x)\cdot f_1(y)=f_1(x\pm y)$$

and $$f_2(x)\cdot f_2(y)\pm f_1(x)\cdot f_1(y)=f_2(x\pm y).$$

16. If $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$, then $f(2x)=\dfrac{2f(x)}{1+\{f(x)\}^2}$,

$$f(x+y)=\frac{f(x)+f(y)}{1+f(x)\cdot f(y)} \text{ and } f(3x)=\frac{\{f(x)\}^3+3f(x)}{1+3\{f(x)\}^2}.$$

17. If $2s=a+b+c$ and $4(s-b)(s-c)=bc$, then

$b^2+c^2=a^2+bc$, $4s(s-a)=3bc$ and $a^4+b^4+c^4=2a^2(b^2+c^2)-b^2c^2$.

18. If $ax^n+by^n+cz^n$ be divisible by $pq(x^2+yz)-(q^2y+p^2z)x$, then $bp^{2n}+cq^{2n}+ap^nq^n=0$.

19. Simplify

$$\frac{(a+p)(a+q)}{(a-b)(a-c)(x+a)}+\frac{(b+p)(b+q)}{(b-c)(b-a)(x+b)}+\frac{(c+p)(c+q)}{(c-a)(c-b)(x+c)}.$$

20. The areas of the cross sections of two water pipes are as 3:5, and the velocities of the water through the pipes are as 3:4. At the end of an hour 1,221 gallons more have flowed through the second than through the first. Find the number of gallons which flow through each pipe per hour.

21 A man walking upon the railway track has partly crossed a bridge whose length is l, when he perceives a train approaching from behind at a distance d from the bridge. His position is such that it is equally safe to advance or to retreat. Find the fraction of the bridge he has crossed, and compare his rate with the rate of the train, providing he has just time to escape the train.

22. Sum $1+(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\ldots.$ to n terms.

23. Sum $(x-y)+\left(\frac{y^2}{x}-\frac{y^3}{x^2}\right)+\left(\frac{y^4}{x^3}-\frac{y^5}{x^4}\right)+\ldots.$ to n terms.

24. Sum to n terms the series whose n^{th} terms are

$2^n.3^{n+1}$, $(x^n+n)(x^n-n)$, $(x^n-x^{-n})(y^n-y^{-n})$, $2^n(3^{n-1}+3^{n-2}+\ldots 1)$.

25. Compare the length of the sides a, b, c of a right-angled triangle, c being the hypotenuse, when the squares described upon them are in harmonical progression.

26. If y be the harmonical mean between x and z, and x and z be the arithmetical and geometrical means respectively between a and b, express y in terms of a and b.

27. Show that a series of numbers in A. P. may be found whose sum to n terms is always an *even* square, but that no such series can always be equal to an odd square.

28. Six men and their wives stand in a row. In how many ways may they be arranged, providing each man must stand beside his wife? In how many ways, providing no man may stand beside his wife?

29. The driver of a four-horse coach can choose his horses from a stable of 6 white and 8 black horses, but he must not pair 2 horses of different colors. In how many ways may he choose his 4 horses?

30. Find the quotient when the sum of all the numbers which can be formed with n significant digits is divided by $\underline{|n-1}$ times the sum of the digits.

31. In a basket are 10 apples at 3 for a cent, and 5 pears at 2 for a cent; a boy has 5 cents in his pocket and wants some fruit; how many choices has he?

32. In how many ways may 6 persons each choose a right and a left-hand glove from 6 pair without any person taking mates?

33. If P_r denote the number of permutations of n things, r at a time, then

$$n(n-1)(P_{n-1}-P_{n-2})(P_{n-2}-P_{n-3})\ldots(P_2-P_1)=P_1 \,.\, P_2 \,.\, P_3 \ldots P_{n-2}P_n.$$

34. Prove that $2^n = 1 + \dfrac{(n+1)n}{\underline{|2}} + \dfrac{(n+1)n(n-1)(n-2)}{\underline{|4}} + \ldots.$

35. Prove that

$$\sqrt{6} = \frac{5}{2}\left\{1 - \frac{1}{\underline{|2}} \cdot \frac{2^3 \,.\, 3}{5^4} - \frac{1 \,.\, 3 \,.\, 5}{\underline{|4}} \cdot \frac{2^6 \,.\, 3^2}{5^8} - \frac{1 \,.\, 3 \,.\, 5 \,.\, 7 \,.\, 9}{\underline{|6}} \cdot \frac{2^9 \,.\, 3^3}{5^{12}} \ldots\right\}.$$

36. Prove that $\dfrac{23}{24} - \dfrac{2}{3}\sqrt{2} = \dfrac{1}{2^3\,\underline{|3}} - \dfrac{1}{2^4} \cdot \dfrac{3}{4} + \dfrac{1 \,.\, 3 \,.\, 5}{2^5\,\underline{|5}} - \ldots.$

37. In the expansion of $(1-x)^{-\frac{1}{n}}$ prove that the sum of the co-

efficients of the first r terms bears to the coefficient of the r^{th} term the ratio of $1+n(r-1)$ to 1.

38. Find the $(r+1)^{\text{th}}$ terms of $(1-4x)^{-\frac{1}{2}}$ and $\dfrac{1}{(a^5-x^5)^{\frac{1}{5}}}$.

39. Find the coefficient of x^{2n} in the expansion of $\dfrac{(1-x)^5}{(1-2x^2)^4}$.

40. Sum to n terms,

$$1+2(2+3)+3(4+5+6)+4(7+8+9+10)+\ldots.$$

41. Show that the coefficient of x^n in $(1-x)^{-n}$ is equal to the sum of all the preceding coefficients.

42. In the expansion of $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}$ the coefficients of the $(2r-1)^{\text{th}}$ and the $(2r)^{\text{th}}$ terms are equal.

43. If $p_r = \dfrac{1.3.5\ldots 2r-1}{2^r \lfloor r}(-1)^r$ and $q_r = \dfrac{1.3.5\ldots 2r-3}{2^r \lfloor r}(-1)^{r-1}$,

then $$p_0+p_1+p_2+\ldots = \frac{1}{2}(q_0+q_1+\ldots.)$$

and $$p_0q_n+p_1q_{n-1}+p_2q_{n-2}+\ldots.=0.$$

44. If x be small, find the value of $\dfrac{1}{1+\sqrt{1-x}} - \dfrac{1}{1+\sqrt[3]{1-x}}$.

45. Solve $\dfrac{x+a^3+b^3+c^3}{x+3abc} = \dfrac{a^2+b^2+c^2}{ab+bc+ca}$.

46. Solve $\dfrac{x+2y}{3z} = \dfrac{y+2z}{3x} = \dfrac{z+2x}{3y} = x+y+z$.

47. The numerator and the denominator of one fraction are each greater by 2 than the corresponding terms of a second fraction. The sum of the fractions is $2\frac{5}{8}$; but if the denominators were interchanged their sum would be 3. Find the fractions.

48. The front wheel of a carriage makes 78 revolutions more than the hind wheel in going 780 yards; but if the circumference

of each were increased 1 yard, the former would make only 50 revolutions more than the latter in going the same distance. Find the circumference of each wheel.

49. Solve
$$x+y+z=1,$$
$$xy+yz-zx=1,$$
$$x^2+y^2-z^2+xy=0.$$

50. Eliminate x, y, z from
$$x=yz\left(\frac{b}{y}+\frac{c}{z}-\frac{a}{x}\right),$$
$$y=zx\left(\frac{c}{z}+\frac{a}{x}-\frac{b}{y}\right),$$
$$z=xy\left(\frac{a}{x}+\frac{b}{y}-\frac{c}{z}\right).$$

51. Solve
$$\sqrt{x}=\sqrt{yz}\left(\frac{b}{y}+\frac{c}{z}-\frac{a}{x}\right),$$
$$\sqrt{y}=\sqrt{zx}\left(\frac{c}{z}+\frac{a}{x}-\frac{b}{y}\right),$$
$$\sqrt{z}=\sqrt{xy}\left(\frac{a}{x}+\frac{b}{y}-\frac{c}{z}\right),$$

52. If $\dfrac{d^2}{D^2}=\dfrac{n}{N}$ where d is the difference of the roots of $x^2+mx+n=0$ and D the difference of the roots of $x^2+Mx+N=0$, then $\dfrac{m^2}{M^2}=\dfrac{n}{N}$.

53. The coefficient of x^r in $(1-ax)^{-2}(1-bx)^{-2}$ is
$$\frac{r+1}{(a-b)^3}(a^{r+3}-b^{r+3})-\frac{(r+3)ab}{(a-b)^3}(a^{r+1}-b^{r+1}).$$

54. Show that the equation, $x^n+rx^{n-p}+s=0$ will have equal roots if $\left\{\dfrac{r}{n}(p-n)\right\}^n=\left\{\dfrac{s}{p}(n-p)\right\}^p$.

55. If the roots of $x^2+px+q=0$ are real, so also are the roots of the equation, $(mp+2)x^2+2(mq+5p)x+18q=0$.

56. Solve $x^4 + 1 = 2(1+x)^4$.

57. Prove $\left(\frac{p}{q}\right)^{\frac{1}{n}} = \frac{p}{q}\left\{\frac{(2n-1)q+p}{(2n-1)p+q}\right\}$ if p is nearly equal to q.

58. If $r<1$ and positive, and m is a positive integer, show that $(2m+1)r^m(1-r)<1-r^{2m+1}$.

59. Show that the sum of the products of the first n natural numbers, three together, is

$$\frac{(n-2)(n-1)n^2(n+1)^2}{48}.$$

60. Find the condition that the equations,

$$lx^2+my^2+nz^2=0,$$
$$ax+by+cz=0,$$

may have only one set of values for the ratios $x:y:z$, and show that if this condition hold,

$$\frac{lx}{a}=\frac{my}{b}=\frac{nz}{c}.$$

61. Sum to n terms and to infinity,

$$\frac{1}{1.6}+\frac{1}{6.11}+\frac{1}{11.16}+\frac{1}{16.21}+\dots.$$

62. The equations, $x^3+y^3+z^3-3xyz=a^3$,

$$yz+zx+xy=b^2,$$
$$x+y+z=c,$$

cannot be simultaneously true unless $c^3-a^3=3cb^2$; and if this holds, they are true for an infinite number of finite values of x, y, z.

63. Show that

$$(1+x+x^2)(1+x^3+x^6)\dots(1+x^{3^{n-1}}+x^{2.3^{n-1}})=1+x+x^2+\dots x^{3^n-1}.$$

64. Prove that the coefficient of x^{2n} in the expansion of

$$\frac{1}{(1-x)(1+x)^4} \text{ is } \frac{(n+1)(4n^2+11n+6)}{6}.$$

65. If x be a positive integer, prove that $\dfrac{1-2x^x+x^{x+1}}{(1-x)^2}$ is a positive integer.

66. If $mx_1^2+ny_1^2=a^2$, $mx_2^2+ny_2^2=a^2$, and $mx_1x_2+ny_1y_2=0$, then $x_1^2+x_2^2=\dfrac{a^2}{m}$ and $y_1^2+y_2^2=\dfrac{a^2}{n}$.

67. If ${}_nP_r$ denote the number of permutations of n things taken r together, and $\Sigma({}_nP)$ denote ${}_nP_1+{}_nP_2+\ldots\ldots{}_nP_n$, show that

$$\Sigma({}_{n+1}P)=(n+1)\{\Sigma({}_nP)+1\}.$$

68. The coefficient of x^r in the expansion of

$$(1+x)(1+cx)(1+c^2x)\ldots\ldots,$$

the number of factors being unlimited and c less than unity, is equal to

$$\frac{c^{\frac{1}{2}r(r-1)}}{(1-c)(1-c^2)(1-c^3)\ldots\ldots(1-c^r)}.$$

69. There are $p+q$ numbers, $\alpha, \beta, \gamma\ldots\ldots$, of which p are even and q odd. Show that the sum of the products, taken 3 and 3 together, of the quantities, $(-1)^\alpha$, $(-1)^\beta$, $(-1)^\gamma\ldots\ldots$, etc.

$$=\frac{1}{6}\{(q-p)^3-3(q^2-p^2)+2(q-p)\}.$$

70. If $\qquad A=a_0+a_1x+a_2x^2+\ldots\ldots a_nx^n$

and $\qquad B=a_0+a_1y+a_2y^2+\ldots\ldots a_ny^n$,

show that when $a_0=a_n$, $a_1=a_{n-1}$, etc., $A:B=x^{\frac{n}{2}}:y^{\frac{n}{2}}$, where x and y are the roots of $x^2+px+1=0$.

71. If x, y, z be three positive quantities whose sum is unity, then will $(1-x)(1-y)(1-z)>8xyz$.

72. If $x+y+z=x^2+y^2+z^2=2$, then will

$$x(1-x)^2=y(1-y)^2=z(1-z)^2.$$

73. The equation,

$$(x+\sqrt{x^2-bc})(y+\sqrt{y^2-ca})(z+\sqrt{z^2-ab})=abc,$$

is equivalent to $ax^2+by^2+cz^2=abc+2xyz$.

74. Prove that the equations,

$$x+y+z=a+b+c,$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

$$\frac{x}{a^3}+\frac{y}{b^3}+\frac{z}{c^3}=0,$$

are equivalent to only two independent equations if

$$bc+ca+ab=0.$$

75. If α, β be the roots of $x^2+x+1=0$, show that

$$\frac{1}{1+x+x^2}=\frac{1}{\alpha-\beta}\left\{\frac{\alpha}{1-\alpha x}-\frac{\beta}{1-\beta x}\right\}.$$

76. If a, b, c be the roots of the equation, $x^3+qx+r=0$, form the equation whose roots are

$$ab+\frac{1}{a+b},\quad bc+\frac{1}{b+c}\ \text{ and }\ ca+\frac{1}{c+a}.$$

77. There are n lines in a plane, no two of which are parallel and no three pass through the same point. Their points of intersection are joined. Show that the number of fresh lines introduced is

$$\frac{1}{8}(n)(n-1)(n-2)(n-3).$$

78. The number of ways in which r things may be distributed among $n+p$ persons so that certain n of those persons may have one at least is

$$(n+p)^r-n(n+p-1)^r+\frac{n(n-1)}{\lfloor 2}(n+p-2)^r-\ldots.$$

79. If the quantities x, y, z be all integral, and satisfy the equations,

$$\frac{a(y^2z^2+1)+y^2+z^2}{yz}=\frac{a(z^2x^2+1)+z^2+x^2}{zx}=\frac{a(x^2y^2+1)+x^2+y^2}{xy},$$

each member of the equations $=a^2-1$, and

$$xyz(xy+yz+xy)=x+y+z.$$

80. If $x=\dfrac{a+n(a-b)}{b+n(a-b)}$, $\dfrac{a-bx}{(1-x)^2}$ will be equal to the sum of the first n terms of its expansion in ascending powers of x; a, b being unequal quantities.

81. The number of combinations of $2n$ things, taken n together, when n of the things, and no more, are alike, is 2^n; and the number of combinations of $3n$ things, n together, when n of the things, and no more, are alike, is

$$2^{2n-1}+\frac{\underline{|2n}}{2(\underline{|n})^2}.$$

82. The number of ways in which p things may be distributed among q persons, so that every one may have one at least, is

$$q^p-q(q-1)^p+\frac{q(q-1)}{\underline{|2}}(q-2)^p-\ldots.$$

83. Prove that

$$1-\frac{n(1+x)}{1+nx}+\frac{n(n-1)}{\underline{|2}}\,\frac{1+2x}{(1+nx)^2}-\frac{n(n-1)(n-2)}{\underline{|3}}\,\frac{1+3x}{(1+nx)^3}+\ldots.=0.$$

84. Factor

$$(a^3+b^3+c^3)xyz+(b^2c+c^2a+a^2b)(y^2z+z^2x+x^2y)$$
$$+(bc^2+ca^2+ab^2)(yz^2+zx^2+xy^2)+(x^3+y^3+z^3)abc+3abcxyz.$$

85. If $x+y+z+w=0$, then

$$wx(w+x)^2+yz(w-x)^2+wy(w+y)^2+zx(w-y)^2$$
$$+wz(w+z)^2+xy(w-z)^2+4xyzw=0.$$

86. An A. P., a G. P., an H. P. have a and b for their first two terms. Show that the $(n+2)^{\text{th}}$ terms will be in G. P. if

$$\frac{b^{2n+2}-a^{2n+2}}{ba(b^{2n}-a^{2n})}=\frac{n+1}{n}.$$

87. A candidate is examined in three papers, to each of which m marks are assigned as a maximum. His total on the three

papers is $2m$. Show that there are $\frac{1}{2}(m+1)(m+2)$ ways in which this may occur.

88. If r be less than n, find the value of

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} - r\frac{\lfloor n-1}{\lfloor r-1 \lfloor n-r} + \frac{r(r-1)}{\lfloor 2} \cdot \frac{\lfloor n-2}{\lfloor r-2 \lfloor n-r} - \dots .$$

89. By comparing the two expansions of $(1+x)^{2n}$ prove

$$\frac{2^r}{\lfloor n-r} + \frac{r(r-1)}{\lfloor n-r+1} \cdot 2^{r-2} + \frac{r(r-1)(r-2)(r-3)}{\lfloor 2 \lfloor n-r+2} \cdot 2^{r-4} + \dots$$

$$= \frac{\lfloor 2n}{\lfloor n \lfloor 2n-r}.$$

90. If a, b, c, d be the roots of the equation,

$$x^4 + 4px^3 + 6qx^2 + 4rx + t = 0,$$

find the value of $\dfrac{x^n}{x-a} + \dfrac{x^n}{x-b} + \dfrac{x^n}{x-c} + \dfrac{x^n}{x-d}$.

91. Form the series whose m^{th} term is $(m-1)(m+1)(2m-1)$, and sum it to n terms.

92. If a_r be the coefficient of x^r in the expansion of $(1+x)^n$, and c_r be the coefficient of x^r in the expansion of $(1+x)^{2n}$, show that, when n is an even number,

$$(c_0^2 + c_2^2 + c_4^2 + \dots) - (c_1^2 + c_3^2 + c_5^2 + \dots) = (a_0^2 + a_1^2 + a_2^2 + \dots).$$

93. Find the sum of the products of the first n natural numbers, taken two at a time, and show that it is the same as one-half the sum to $n-1$ terms of the series,

$$1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \dots .$$

94. Solve (1) $x^3 - (a-1)x^2 - \left(a + \dfrac{1}{a}\right)x + 1 = 0$;

(2) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$.

95. Show that

$$n+\left(\frac{n}{\underline{|1}}\right)^2\frac{n-1}{2}+\left\{\frac{n(n-1)}{\underline{|2}}\right\}^2\frac{n-2}{3}+\left\{\frac{n(n-1)(n-2)}{\underline{|3}}\right\}^2\frac{n-3}{4}+\ldots$$

$$=\frac{2n(2n-1)(2n-2)\ldots\ldots(n)}{\underline{|n+1}}.$$

96. If $a+b+c=0$, prove that $\dfrac{a}{bc-a^2}+\dfrac{b}{ca-b^2}+\dfrac{c}{ab-c^2}=0.$

97. Having given, for all values of n, the relation,

$$a_1a_2a_3\ldots\ldots a_n=a_1^{\,n^2},$$

find the sum, to n terms, of the series $a_1+a_2+a_3+\ldots\ldots a_n$.

98. If the roots of $x^3+px^2+qx+r=0$ be in A. P., prove

$$2p^3-9pq+27r=0.$$

99. The equation, $x^4+4x^3+3x^2-2x-6=0$ has one root of the form $-1+\sqrt{-1}$. Find all the roots.

100. If a, b, c are real quantities, prove that no real values of x and y can satisfy the equation, $ay-bx=c\sqrt{(x-a)^2+(y-b)^2}$, unless c^2 is less than a^2+b^2.

101. Write down all the numbers that can be composed of the four digits, 3, 4, 5, 6, which are divisible by 11.

102. Six papers are to be set in an examination, two of them in Mathematics. In how many different orders may the papers be given, provided only that the two mathematical papers do not come together?

103. Find the sum of the series,

$$\frac{4}{1\,.\,5}+\frac{9}{5\,.\,14}+\frac{16}{14\,.\,30}+\frac{25}{30\,.\,55}+\ldots\ldots \text{ to } n \text{ terms},$$

the last factor in the denominator being the sum of the other factor and the numerator.

104. If n is greater than 1 in the series,

$$\frac{1}{n} + \frac{2}{n^2} + \frac{3}{n^3} + \frac{4}{n^4} + \dots,$$

show that the sum to infinity is $\dfrac{n}{(n-1)^2}$, and the sum to m terms,

$$\frac{n(n^m-1) - m(n-1)}{n^m(n-1)^2}.$$

105. If

$$\left.\begin{aligned} \frac{x-a}{b} + \frac{y-b}{c} + \frac{z-c}{a} &= 0, \\ \frac{x-b}{c} + \frac{y-c}{a} + \frac{z-a}{b} &= 0, \\ \frac{x-c}{a} + \frac{y-a}{b} + \frac{z-b}{c} &= 0, \end{aligned}\right\}$$

show that

$$x = y = z = \frac{ab^2 + bc^2 + ca^2}{ab + bc + ca}.$$

106. Show that $\dfrac{(b-c)(b+c)^3 + \text{anal.} + \text{anal.}}{(b-c)(b+c)^2 + \text{anal.} + \text{anal.}} = 2(a+b+c)$.

107. Show that $n^{-n} = \left\{ \dfrac{1}{\underline{|2}} - \dfrac{2n-1}{\underline{|3}} + \dfrac{(2n-1)(3n-2)}{\underline{|4}} - \dots \right\}^{n-1}$.

108. If a, b, c are the roots of $x^3 - px^2 + qx - r = 0$, express

$$\frac{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}{2ab + 2bc + 2ca - a^2 - b^2 - c^2}$$

in terms of p, q and r.

109. If $x(x-1)^2 + y(y-1)^2 + z(z-1)^2 - yz(x+1)^2 - zx(y+1)^2 - xy(z+1)^2 + 4xyz = 0$, and $x + y + z > 1$, prove that

$$(x+1)(y+1)(z+1) = x^2 + y^2 + z^2 + 1.$$

110. If $x^4 + px + q$ be divisible by $x^2 + ax + b$, then will

$$(a^3+p)(a^3-p) = 4a^2q \quad \text{and} \quad (b^2+q)(b^2-q) = p^2b^3.$$

111. If $f(x)$ be divided by $x-a$, and the integral quotient by $x-b$, and the second quotient by $x-c$, the remainder will be

$$-\frac{(b-c)f(a)+(c-a)f(b)+(a-b)f(c)}{(b-c)(c-a)(a-b)}.$$

112. If $\dfrac{a+cx}{c+ax}$ be expanded in series ascending by powers of $(1-x)$ and $(1+x)$, and A and B be the coefficients of $(1-x)^n$ and $(1+x)^n$ respectively, then

$$\frac{A}{B}=(-1)^n\left(\frac{c-a}{c+a}\right)^{n+1}.$$

113. On a railway there are 20 stations. Find the number of tickets required in order that a person may travel from any one station to any other.

114. Show that if $\dfrac{a^2}{2b\sqrt{2a^2-b^2}}<1$, a must be $<b(1+\sqrt{3})$.

115. Having given the equations,

$x+y+z=0,\ \ x_1+y_1+z_1=0,\ \ a^2=x^2+x_1^2,\ \ b^2=y^2+y_1^2,\ \ c^2=z^2+z_1^2,$

prove that $\quad a^2(yz-y_1z_1)+b^2(zx-z_1x_1)+c^2(xy-x_1y_1)=0.$

116. Show that if $a+b+c=0$, then

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)=9.$$

117. Prove that if

$$\frac{1}{1+l+ln}+\frac{m}{1+m+ml}+\frac{mn}{1+n+mn}=1,$$

and
$$\frac{l}{1+l+ln}+\frac{ml}{1+m+ml}+\frac{1}{1+n+mn}=1,$$

and none of the denominators be zero, then $l=m=n$.

118. There are n letters and n directed envelopes. In how many ways could all the letters be put into the wrong envelopes?

ANSWERS AND RESULTS.

EXERCISE I. (PAGE 17.)

1. na^{n-1}. 2. na^{n-1}, ∞. 3. 3. 4. ∞, 0.

5. 0, 3, ∞. 6. $\dfrac{\sqrt{2a}}{a\sqrt{3}+1}$. 7. $\dfrac{1}{2\sqrt{a}}$. 8. $\dfrac{-c}{b}$ or ∞.

9. 0, ∞; ∞, ∞; a, ∞, according as $a >, =, <, 1$.

10. No definite value. 11. $\pm\dfrac{1}{2(a-b)}$.

EXERCISE II. (PAGE 23.)

1. 25 : 49; 17 : 20. 2. 10 : 11; $x+1 : x+2$; $a^3+b^3 : a^2+b^2$;

3. 3 : 5, ratio of equality. 4. 495 and 693.

5. (1) $\dfrac{b+d}{a-c}$; (2) 3 or $-\dfrac{2}{3}$; (3) 2 or $\dfrac{1}{2}$; (4) $\dfrac{n(a+1)}{m(a-1)}$.

6. 35 and 65. 7. \$2.40. 8. 929, 260.

9. $\dfrac{ad-bc}{c-d}$. 15. The latter. 16. $\dfrac{nq-mp}{mq+np+2mp}$.

17. $\dfrac{npt}{mq-np}$, $\dfrac{mqt}{mq-np}$. 19. 56, 7 : 8. 20. 2 : 3 and 4 : 5.

21. $mq : np$. 22. $n^2qr : m^2ps$. 23. 6 min., 4 min.

24. $\dfrac{a(p+q)(r+s)}{qr \sim ps}$. 25. $q+r-p : p+q-r$.

EXERCISE III. (PAGE 31.)

1. $1 : -2 : 1$. 2. $b(1+a) : b(1+c) : 1-ac$.

3. $x = -y = z = \pm 2\sqrt{21}$. 4. $x = 10$, $y = 16$, $z = 7$.

10. $x=a-b,\ y=b-c,\ z=c-a$. 11. $x=b+2c,\ y=c+2a,\ z=a+2b$.

12. $x=b+c-a$, $y=c+a-b$, $z=a+b-c$.

13. $x=b$, $y=c$, $z=a$.

14. $x=\dfrac{b-c}{a}$, $y=\dfrac{c-a}{b}$, $z=\dfrac{a-b}{c}$.

15. $x=5$, $y=4$, $z=3$.

16. $x^3=\dfrac{b^2c^2}{a}$, $y^3=\dfrac{c^2a^2}{b}$, $z^3=\dfrac{a^2b^2}{c}$.

18. $abc+2fgh-af^2-bg^2-ch^2=0$. 19. $a^3+b^3+c^3-3abc=0$.

21. $2abc+ab+bc+ca=1$.

EXERCISE IV. (Page 41.)

1. 25. 2. 8. 3. $7+5\sqrt{2}$, $\sqrt{7}+\sqrt{5}$. 4. 7.

5 7 or -19. 6. $m\sqrt{\dfrac{a}{b}}$, $m\sqrt{\dfrac{b}{a}}$. 7. 3, 12, 48, or $13\frac{1}{2}$, $22\frac{1}{2}$, $37\frac{1}{2}$.

8. 9 or 11. 12. $\dfrac{ab}{a+b}$. 13. $\dfrac{2p}{s-d}$, $\dfrac{2p}{s+d}$. 16. $\dfrac{bc-ad}{a-b-c+d}$.

24. \$40 or \$60; \$50. 25. 30, 45. 26. 7, 8, 9.

27. $3\frac{1}{3}$. 28. $(m+n)(qr-ps):(p+q)(ms-nr)$. 31. 64, 36.

32. $\dfrac{(2a-b)c}{a+b}$, $\dfrac{(2b-a)c}{a+b}$; $\dfrac{3ac}{a+b}$, $\dfrac{3bc}{a+b}$.

EXERCISE V. (Page 54.)

1. 45. 2. 10. 3. $10:1$. 4. $2:3$.

5. 10 or 15. 6. $\dfrac{7}{5}$. 7. $\dfrac{b^2}{a}$. 8. $s=\dfrac{1}{2}ft^2$.

9. 6, $18:25$. 11. n^2a. 12. ± 16. 22. 143.

23. $4:5$. 24. $244\frac{4}{9}$ ft. ·000906 in., nearly.

25. $\dfrac{a-2b+c}{2}$ dollars. 26. $\dfrac{mcn^2}{(m+1)a^2}$; $\dfrac{cn^{\frac{3}{2}}}{(m+1)b^{\frac{3}{2}}}$.

27. 5.376 cwt. 28. \$93.75.

29 $224\frac{1}{2}$ days, nearly. 30. 1 day 18 hrs. 28 min.

EXERCISE VI. (Page 63.)

1. 23, 33, 203. 2. 14, -7, $38-3n$.
3. (1) $3n+1$; (2) $11-3n$; (3) $1\frac{1}{3}$, 0; (4) 37, $5n-33$.
4. 34th, 46th, no term. 5. 7, 290. 6. 194.
7. $(n+1)a+(n-4)b$, $2na+(2n-5)b$, $(3p+4)^{\text{th}}$.
8. (1) 5050; (2) -790; (3) -333; (4) n^2; (5) $1325\sqrt{3}$; (6) $-n$.
9. 25, 16, m. 10. 5, 8, 11, 14, 17. 11. $x, 2x-1 \ldots x^2-x+1$.
12. $-(m^2+mn+n^2)$. 13. 7, 4. 14. 8, $8n-14$. 15. 59.
16. n^2. 17. 110. 18. $\dfrac{(m+2)(a+b)}{2}$. 19. c. 21. 7.
22. 6, 11, 16. 23. 3, 5, 7. 24. 20. 25. n^2.
26. 8, 10, 12, 14. 27. $\pm1, \pm3, \pm5$; $6\sqrt{\dfrac{2}{7}}$, $5\sqrt{\dfrac{2}{7}}$, $4\sqrt{\dfrac{2}{7}}$.
28. $\dfrac{1}{10}, \dfrac{2}{10}, \dfrac{3}{10}, \dfrac{4}{10}$. 29. $\dfrac{pq+1}{2}$. 30. 3 or 8 hours.
34. $p(2-q)$, $\dfrac{2p(q-1)}{n-1}$. 35. $\dfrac{(p-2)(2a-d)}{d(4-p)}$.
36. $\dfrac{n(n-1)}{2}+1$, $\dfrac{n(n+1)}{2}$, $\dfrac{n(n^2+1)}{2}$, $\dfrac{n(n+1)(n^2+n+2)}{8}$.
38. $\dfrac{n^2(n^2+1)}{2}$.

EXERCISE VII. (Page 72.)

1. 11 or 15. 2. 2, $n(n-2)$. 3. $\dfrac{(13-n)(2n+1)}{2}$.
4. $\dfrac{(2n+1)(ma-nb)}{a-b}$. 5. 91, r^2-r+1, n^2+n+1. 6. $10n-8$, 10.
7. 25 or $-\dfrac{76}{3}$. The sum of 76 terms of 0, $\dfrac{1}{3}, \dfrac{2}{3} \ldots$ is 950.
8. $2\frac{5}{8}$. 9. 9 or -11. The sum of 11 terms of 19, 17 .... is 99; the sum of 11 terms of 1, -1, -3 .... is -99.

11. $\frac{2a}{d}$ is a negative integer.

13. $3:2,\ -4:5,\ \frac{10-7r}{3r-1}$.

15. $\frac{m+n}{2},\ \frac{(2q-p)m+pm}{2q}$.

16. $\frac{1}{3},\ \frac{1}{2},\ \frac{2}{3}\ldots$

20. $(n-1)s$.

21. $\frac{n^2(n+1)^2}{4}$.

25. $\frac{(n-q)P-(n-p)Q}{p-q}$.

30. $1 : ab+bc+ca : a+b+c$.

35. $1, 3, 5, 7\ldots$

39. $n^{p-1}-n+1$.

40. $\left\{\frac{n(n^{p-2}+1)}{2}\right\}^2-\left\{\frac{n(n^{p-2}-1)}{2}\right\}^2$.

EXERCISE VIII. (PAGE 80.)

1. 32.

2. 19683.

3. $\frac{1}{256}$.

4. $2(3)^{n-1}$.

5. 2^{2n}.

6. $a^n r^{2n-2}$.

7. $-\frac{b^{2n-2}}{a^{2n-3}}$.

8. $(2n-1)a^{n-1}r^{n-2}$.

9. $\frac{\sqrt{3}}{2}(\sqrt{3}-1)^n$.

10. $\frac{(\sqrt{2}-1)^{2n}}{2^{n+1}}$.

11. $1023,\ 2^n-1$.

12. $49\frac{49}{1215},\ \frac{12}{5}\left\{\left(\frac{5}{3}\right)^n-1\right\}$.

13. $\frac{1-2^{2n}}{3}\cdot\frac{1+2^{2n+1}}{3}$.

14. $\frac{1023}{1024}\cdot 1-\frac{1}{2^n}$.

15. $\frac{a^2\{1-(-ax)^n\}}{x(1+ax)},\ \frac{a^2\{1+(ax)^{2n-3}\}}{x(1+ax)}$.

16. $\frac{(2+\sqrt{3})^n+\sqrt{3}-2}{\sqrt{3}+1}$.

17. $\frac{a(r^n-1)}{r^2(r-1)}$.

18. $\frac{b(a^{n+3}-b^{2n+6})}{a^{n+2}(a-b^2)}$.

19. $\left(\frac{3}{2}\right)^n-\frac{2}{3}$.

20. $\frac{2^{\frac{5-n}{4}}(2^{\frac{n}{4}}-1)}{2^{\frac{1}{4}}-1},\ \frac{15}{8-2^{\frac{11}{4}}}$.

21. $\frac{1}{2}\left\{1-\left(-\frac{2}{3}\right)^n\right\},\ \frac{1}{2}\left\{1+\left(\frac{2}{3}\right)^{2n+1}\right\}$.

22. ± 4.

23. $3,\ 3(2)^{n-1}$.

24. 5^{th}.

25. $\frac{1}{3},\ 2,\ 12$.

26. $1, 3, 9$.

27. $\frac{1}{2}\{m+\sqrt{m^2-4n^2}\},\ \frac{1}{2}\{m-\sqrt{m^2-4n^2}\}$.

28. $12, 16$.

29. $\frac{4}{9},\ \frac{2}{3}$, etc.

30. 2.

31. $1,\ \frac{3}{2}$, etc., or $\frac{19}{3},\ -\frac{19}{6}$, etc.

32. $2\frac{1}{2}$, 5, etc., or $-7\frac{1}{2}$, 15, etc. 33. $\frac{1}{17}, \pm\frac{4}{17}, \frac{16}{17}, \pm\frac{64}{17}$.

34. 2, 4, 8, 12, 16, or $\frac{216}{17}$, $-\frac{216}{9}$, etc.

35. 5, 9, 13, or 23, 9, -5. 36. 5, 10, or $-8\frac{1}{3}$, $-323\frac{1}{3}$.

37. $2^{n-2}(2^n + 2^{n-1} - 1)$, $2^{n-1}(2^n - 1)$.

38. $2^{n+1} - 3$, $3 \cdot 2^{2n-2} - 2^n$, $(2^n - 1)^2$.

39. $2^{\frac{n(n-1)}{2}}(2^n - 1)$, $2^{\frac{n(n+1)}{2}} - 1$. 40. $2^{2^n-1} - 1$, $\frac{1}{2}\{2^{2^n} - 2^{2^{n-1}}\}$.

42. $\frac{a^2(1-r)(1-r^{2n})}{1+r}$. 43. $\frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, $\frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

EXERCISE IX. (PAGE 87.)

1. 1. 2. $\frac{1}{3}$. 3. $\frac{1}{2}$. 4. $\frac{1}{2}$. 5. $\frac{8}{21}$. 6. $2(2+\sqrt{2})$.

7. $\frac{2\sqrt[3]{2}}{\sqrt[3]{2}+1}$. 8. $\frac{5+3\sqrt{3}}{2}$. 9. $\frac{3(3\sqrt{2}-2)}{14}$. 10. $\frac{a\sqrt{ab}}{\sqrt{ab}-1}$.

11. $4+3\sqrt{2}$. 12. $\frac{3+\sqrt{3}}{2}$. 13. $\frac{nx^n}{x-1} - \frac{x^n-1}{(x-1)^2}$; $\frac{1}{(1-x)^2}$.

14. $4-(n+2)2^{-n+1}$, 4. 15. $6-\frac{2n+3}{2^{n-1}}$, 6. 16. $\frac{2}{9}+\frac{6n+1}{9(-2)^{n-1}}$, $\frac{2}{9}$.

17. $\frac{ar(1-b^nr^n)}{(1-r)(1-br)} - \frac{a(1-b^n)r^{n+1}}{(1-r)(1-b)}$, $\frac{ar}{(1-r)(1-br)}$. 19. c^{2n+1}.

20. $(ab)^{\frac{n+2}{2}}$. 21. $(mn)^{\frac{1}{2}}$, $m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$. 22. $\left(\frac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}$.

30. $\frac{\sqrt{6}}{\sqrt{3}-\sqrt{2}}$. 31. $\frac{1}{\sqrt{2}}$. 32. $\frac{10}{27}(10^n-1)-\frac{n}{3}$.

33. $a=\frac{m(n+1)}{n+2}$, $r=\frac{1}{n+1}$.

34. $\frac{ar(r^n-1)}{(r-1)^2} - \frac{an}{r-1}$, $\frac{ar^n(r^{mp}-1)}{(r-1)(r^p-1)} - \frac{am}{r-1}$.

40. $\frac{a^2}{a-b}$, $\frac{a^4b}{2(a^2-b^2)^{\frac{3}{2}}}$. 41. $\left(\frac{\sqrt{5}+1}{2}\right)A$, A.

EXERCISE X. (PAGE 97.)

1. $-\frac{12}{5}$. 2. $\frac{4}{7-n}$. 3. 1. 4. 17, ± 8, $3\frac{13}{17}$. 5. $\frac{6}{5}, \frac{6}{4}$.

6. $\frac{64}{7}, \frac{32}{5}, \frac{64}{13}$. 7. $\frac{1}{4}, \infty, -\frac{1}{4}, \frac{1}{6-2n}$.

8. $\frac{ab}{(n-1)a-(n-2)b}$. 9. $a+b$. 10. 3, 1.

11. $3\frac{3}{5}$. 12. 14 or $\frac{2}{7}$. 13. $20\frac{1}{4}$, 4.

14. $-\frac{1}{2}, \infty, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}$, or $\frac{5}{24}, \frac{5}{17}, \frac{1}{2}, \frac{5}{3}, -\frac{5}{4}, -\frac{5}{11}, -\frac{5}{18}$.

15. 104, 234. 16. Half the middle term. 17. 2, 3, 6.

18. a, b, c. 24. $\frac{2ar(r^n-1)}{r^2-1}$. 27. $bc(q-r)+ca(r-p)+ab(p-q)=0$.

28. $\frac{2mn}{m+n}, \frac{2mnq}{2nq+(m-n)p}$. 34. No.

EXERCISE XI. (PAGE 107.)

1. 2870. 2. 4960. 3. 3920. 4. 6214.
5. 4466. 6. 5915. 7. 4944. 8. 220, 385.
9. 1375. 12. 330. 13. 4970. 14. 20, 15.
15. 120. 16. 50. 17. 20540. 18. 2024, 3795.
19. 1981, 25.

EXERCISE XII. (PAGE 109.)

1. $\frac{2n(n+1)(2n+1)}{3}$. 2. $\frac{n(4n^2-1)}{3}$.

3. $\frac{n(n-1)(2n-1)}{6}+n(n-1)a+na^2$. 4. $n^2(2n^2-1)$.

5. $\frac{n(n+1)(n+2)(3n+1)}{12}$. 6. $\frac{n(n+1)(n+2)}{6}$.

7. $(-1)^{n+1}\cdot\frac{n(n+1)}{2}$. 8. $\frac{1}{2}\{(4n^2-1)(-1)^{n+1}-1\}$.

9. $n(2n^2-3n-1)$. 10. $\frac{n(n+1)(n+2)}{6}$, $\frac{np(n+1)}{12}\{(2n+1)p+3\}$.

11. $\frac{n(5n^2+1)}{6}$. 12. $\frac{2}{3}$. 13. $\frac{5}{8}$, 1, $\frac{8}{5}$ or $-\frac{3}{8}$, -1, $-\frac{8}{3}$.

14. 6, $\frac{5}{54}$. 15. 2^n+1, $2^{n+1}+2$, etc.

16. $\frac{n(n-1)(n+1)(3n+2)}{24}$. 20. $\frac{n(n+1)(2n^2+2n-1)}{2}$.

21. $-3\frac{1}{2}$. 22. $\frac{n(n+1)}{12}\{6a+(4n-1)d\}$.

23. $\frac{1}{8}\{(-1)^{n+1}(4n^3+6n^2-1)-1\}$.

25. $-n(8n+9)$, $\frac{1}{4}\{7+(-1)^{n+1}(8n^2+18n+7)\}$.

33. 9, 15, 25. 39. $\frac{n(2n+1)(a-b)^2}{6(n+1)b}$.

46. $\frac{n}{12}(n+4)(2n+1)$, $\frac{n+1}{12}(2n^2+n+3)$, $\frac{n}{2}\{n+1+(-1)^n(n-1)\}$.

47. $2(n^2+n)$, $\frac{n(n+1)(n+5)}{3}$; $2n$, $\frac{n(n+1)(n+2)}{3}$;

$\{(-1)^n+1\}n^2+2n$, $\frac{n(n+1)}{6}\{2n+7+(-1)^n3\}$

49. $\frac{n^2(n+1)^2}{2}$.

EXERCISE XIII. (PAGE 118.)

1. 78117. 2. 1011102. 3. 12710442. 4. 4776362.
5. $2ee008\frac{8}{11}$. 6. $12121\frac{103}{1323}$. 7. 103466023. 8. 1045.
9. $2t8$. 10. 1783661. 11. 5647124.
12. 51215405. 13. 3400. 14. 1101111.
16. $2^9+2^8+2^7+1$, $2^{10}+2^8+2^7+2^6+2^5+2^4+2+1$.
17. $3^7+1-(3^5+3^4+3^2+3)$. 18. Nonary. 19. Undenary.
20. Septenary. 22. Octenary. 23. Octenary.
24. $3\frac{5}{60}$. 25. 140 sq. ft. $\frac{1}{3}$ sq. in. 26, 11 ft. 9 in,

EXERCISE XIV. (PAGE 121.)

1. ·33, ·5343, ·74, ·e3. 2. ·875, ·8$\dot{3}$, ·42857$\dot{1}$.

3. ·05343, ·113. 4. ·9167406.... 5. 12t96·515228.

6. $\dfrac{67}{300}$, $\dfrac{21}{50}$. 7. 1, $\dfrac{3}{14}$, $\dfrac{5}{6}$, $\dfrac{9}{10}$. 8. 20110004·3.

9. 1209·$\dot{t}t$183654729$\dot{0}$, 10t·t09349787026....

10. Six and twelve. 11. Two. 12. Nine.

15. Last digit 4 or 0; last digit 0 and preceding digit even.

16. 36. 27. ·3125.

31. The given multipliers must be replaced by 1, 10, 9, 12, 3, 4.

EXERCISE XV. (PAGE 140.)

1. 3·1622776601, ·8660254037, 1·7724538509.

2. 1·2599210498, ·5848035476, 1·2407009818.

3. ·9510565, ·2588190. 4. $\sqrt[4]{18}+\sqrt[4]{50}$, $2\sqrt[4]{3}-\sqrt[4]{27}$, $\sqrt[4]{2}-\sqrt[4]{\dfrac{1}{2}}$.

5. $\sqrt{5}+\sqrt{7}-2$, $\sqrt{2}+\sqrt{3}+\sqrt{5}+\sqrt{30}$.

6. $3-\sqrt{7}+\sqrt{2}-\sqrt{3}$. 7. $1+\sqrt{2}$, $3-\sqrt{5}$, $3\sqrt{3}-\sqrt{6}$.

8. 0. 9. $2\sqrt{3}-3$, $1-x+\sqrt{1-x}$. 10. $x^4+2x^3-8x^2-6x-1$.

11. -3. 12. 44. 13. $\dfrac{1}{2}(\sqrt{2}+\sqrt{3}+\sqrt{5})$.

15. $\dfrac{x}{\sqrt{x^2+y^2}}$. 16. $\sqrt[3]{2}$. 18. $x^{12}+1$.

19. $x^3+3x\sqrt[3]{2}+1$. 20. $x^2\sqrt[3]{4}-x(\sqrt[3]{2}+2)+\sqrt[3]{16}-\sqrt[3]{4}+1$.

21. $\dfrac{1}{5}\sqrt{5}$, $\sqrt{6}-\sqrt{3}+\sqrt{2}-1$. 22. $7\frac{1}{4}$. 24. $\sqrt{3}-\sqrt[4]{7}$, $\sqrt[3]{12}$.

25. 2. 26. $2r$. 27. 0. 28. $\dfrac{1}{2}(\sqrt{6}-2\sqrt{2}+\sqrt{3}-1)$, $\dfrac{1}{3}\sqrt{3}$.

30. $3a+\sqrt{b^2-3a^2}$. 33. 5·23607.

35. $\pm 2(\sqrt{3}-\sqrt{2})$ or $\pm 2(\sqrt{6}-2)$. 36. $\dfrac{1}{\sqrt{2}}$, $\dfrac{13}{90}$.

37. $\left(\dfrac{a+b}{a-b}\right)^{\frac{q+p}{q-p}}$.

EXERCISE XVI. (PAGE 164.)

1. $205, -102i$. 2. $8\sqrt{3}$. 3. $-46-9i, -4-4i$.

4. $-44, 8i-2(\sqrt{5}+\sqrt{3})$. 5. $\frac{18}{13}, i\sqrt{3}+\sqrt{2}$.

6. $3-2i, 2-i\sqrt{3}, 2+i\sqrt{5}$. 7. $\sqrt{3}-i\sqrt{2}, 2+i\sqrt{3}$.

8. $27i$. 10. $x^4-6x^3+18x^2-26x+21$. 11. $2-i\sqrt{3}$.

12. $\frac{1}{7}$. 13. $5, m^2+n^2, 1$. 14. $2+(\sqrt{3}-4\sqrt{5})i$.

15. $-\frac{3+i}{\sqrt{2}}$. 16. $\frac{5}{34}$. 17. $-(a+b)(a^2+b^2)$.

18. $abc-bc^2-ca^2-ab^2+i(a^2b+b^2c+c^2a-abc)$.

21. $\frac{1}{2}(n+2-ni), \frac{1}{2}(n+2)(i-1)$.

22. $2abc-\{a^3+b^3+c^3+(a+b)(b+c)(c+a)\}i$. 24. $\frac{1}{\sqrt{1+x^2}}$.

26. $(1+\omega^2)^2, (2+\omega^2)^2$. 27. $-1, 4$. 28. $2\pm\sqrt{3}, 0$.

30. $\omega^2-\omega$ or $i\sqrt{3}$, $\frac{\omega-\omega^2}{3}$ or $\frac{-i\sqrt{3}}{3}$, $\omega^2-2i-\omega$ or $i(\sqrt{3}-2)$.

34. x^6+1. 35. $\frac{3(a^2-bc)}{a^3+b^3+c^3-3abc}$.

EXERCISE XVII. (PAGE 182.)

1. $x=a, b$. 2. $x=\frac{a+2b}{3}, \frac{2a+b}{3}$. 3. $x=5, -3\frac{1}{2}$.

4. $x=0, \frac{7}{12}$. 5. $x=\pm 2, \pm 2\sqrt{-1}$. 6. $x=3, 7, 5\pm\sqrt{11}$.

7. $x=\frac{5a+3b}{8}, \frac{5b+3a}{8}$. 8. $x=\frac{a^2+b^2}{a+b}$ (a simple equat'n).

9. $x=9, -7, 1\pm 2\sqrt{-6}$. 10. $x=-\frac{3}{2}, -\frac{3}{2}, \frac{-3\pm\sqrt{10}}{2}$,

11. $x=\pm\frac{a^2b^2+b^2c^2+c^2a^2}{2abc}$. 12. $x=3, -\frac{2}{3}$. 13. $x=\pm\sqrt{2}$.

14. $x=\frac{1}{2}\{m^2 \pm m\sqrt{m^2+4a}\}$ where $m=\frac{1}{2}\{-a \pm \sqrt{a^2+4b}\}$.

15. $x=\frac{49 \pm \sqrt{97}}{8}$.

16. $x=a\left\{\frac{(1 \pm \sqrt{5})^m - 2^m}{(1 \pm \sqrt{5})^m + 2^m}\right\}$.

17. $x=\frac{7}{9}$.

18. $x=4, -\frac{10}{3}$.

19. $x=6, -\frac{5}{2}$.

20. $x=1, \frac{47-44\sqrt{6}}{23}$.

21. $x=0, \frac{-4(a+b)(a^2+b^2)}{(3a+b)(a+3b)}$.

22. $x=0, 2(1 \pm \sqrt{2})$.

23 $x=\pm 1, \pm\sqrt{\frac{a^2-1}{a^2+3}}$.

24. $x=1 \pm \sqrt{19}$.

25. $x=-1, -\frac{1}{2}\{p-1 \mp \sqrt{(p-3)(p+1)}\}$.

26. $x=\frac{-3+\sqrt{-11}}{2}$.

27. $x=3, -1, 1 \pm 2\sqrt{-1}$.

28. $x=-1, -1, \frac{-3 \pm \sqrt{5}}{2}$.

29. $x=0, 2$.

30. $x=-4, -4$.

31. $x=3, -3$.

32. $x=3, \frac{1}{3}, \frac{-1 \pm \sqrt{-35}}{6}$.

33. $x=0, \frac{63a}{65}$.

34. $x=5, -4, \frac{1 \pm \sqrt{-75}}{2}$.

35. $x=-\frac{1}{3}, \frac{1 \pm \sqrt{-31}}{6}$.

36. $x=\pm 1$.

37. $x=3a-2b, 3b-2a$. Other values are imaginary.

38. $4(a-x)(x-b)=(a-b)^2 \pm \sqrt{(a-b)^4 - 8c(a-b)}$.

39. $x=3, 4, -1$.

40. $x^4 - x^3\left(\frac{a+4b}{a-b}\right) + x^2\left(\frac{a-6b}{a-b}\right) - x\left(\frac{a+4b}{a-b}\right) + 1 = 0$. A reciprocal equation.

41. $\left(x+\frac{c}{ax}\right)^2 + a\left(x+\frac{c}{ax}\right) + b = \frac{2c}{a}$.

42. $x=\pm 3$.

43. $x=(b^{\frac{1}{3}} - a^{\frac{1}{3}})^3$.

44. $x=\left\{\dfrac{a^{\frac{r}{s}}+1}{a^{\frac{r}{s}}-1}\right\}^{\frac{1}{(m+n)^2}}$.

45. $x=\left(\dfrac{a\pm b}{a\mp b}\right)^{\frac{2pq}{q-p}}$.

46. $x=(a^{-\frac{1}{2}}+b^{-\frac{1}{2}})^{\frac{4mn}{m-n}}$.

47. $x^2=\dfrac{1\pm\sqrt{5}}{2}$.

48. $x=2,\ -1,\ \dfrac{5\pm\sqrt{17}}{2}$.

49. $x=-1\pm\sqrt{3}$.

50. $x=\dfrac{m\pm\sqrt{6+2m}}{2}$ where $m=\pm\sqrt{2}$.

51. $x=2\pm\sqrt{2},\ 3\pm\sqrt{3}$.

52. $x=\dfrac{1}{24}\{5\pm\sqrt{5\pm2\sqrt{5}}\}$.

53. $x=\left(-\dfrac{9}{4}\right)^4,\ 3^4,\ \left(\dfrac{3}{2}\right)^4$.

54. $x=a-\dfrac{1}{a},\ \dfrac{1}{2}\left\{-\left(a-\dfrac{1}{a}\right)\pm\sqrt{-3\left(a+\dfrac{1}{a}\right)^2}\right\}$.

55. $x=0,\ -1,\ -1,\ \dfrac{-1\pm\sqrt{-3}}{2}$.

56. $x=2,\ 3,\ \dfrac{-3\pm\sqrt{-15}}{2}$.

57. $x=-1,\ 5,\ 2\pm\sqrt{5}$.

58. $x^2(a-c)+x(a^2+b^2+c^2+ab-3bc)+(a+b)(b+c)(c+a)-6abc=0$.

59. $x=\dfrac{5\pm\sqrt{-11\pm4\sqrt{105}}}{2}$.

60. An identity.

61. $x=0,\ x^2=\pm\sqrt{3}$.

63. $x=3(1\pm\sqrt{3}),\ 1\pm\sqrt{-5}$. (Apply formula of 62.)

64. $x=12,\ -6,\ 2\pm2\sqrt{-11}$.

65. $x=5,\ \dfrac{-5\pm\sqrt{-3}}{2}$.

66. $x=10,\ -5\pm7\sqrt{-3}$.

67. $x=11,\ 11,\ -7$.

68. $x=-7,\ \dfrac{-11\pm\sqrt{389}}{2}$.

69. $x=\dfrac{3}{2},\ \dfrac{-3\pm3\sqrt{5}}{4}$.

EXERCISE XVIII. (PAGE 193.)

1. $x=4,\ 1;$
 $y=1,\ 4.$

2. $x=2,\ 7,\ \dfrac{1}{2}(9\pm\sqrt{-511});$
 $y=7,\ 2,\ \dfrac{1}{2}(9\mp\sqrt{-511}).$

3. $x=7,\ -3,\quad 2\pm\sqrt{-33}$; $y=3,\ -7,\ -2\pm\sqrt{-33}$.

4. $x=5,\ -2,\quad \frac{1}{2}(3\pm\sqrt{-103})$; $y=2,\ -5,\ -\frac{1}{2}(3\mp\sqrt{-103})$.

5. $x=3,\ 2,\ \frac{1}{2}(-7\pm\sqrt{-23}),\ \frac{1}{2}(2\pm\sqrt{22})$;

$y=2,\ 3,\ \frac{1}{2}(-7\mp\sqrt{-23}),\ \frac{1}{2}(2\mp\sqrt{22})$.

6. $x=3,\ -\frac{1}{4}(3\mp\sqrt{-79})$; $y=2,\ -\frac{1}{4}(7\mp\sqrt{-79})$.

7. $x=\pm 3,\ \pm 2$; $y=\pm 2,\ \pm 3$.

8. $x=9,\ 1$; $y=1,\ 9$.

9. $x=5,\ -5$; $y=2,\ -3$.

10. $\frac{x}{y}=-\frac{1}{a^4-b^4}\{a^4+b^4+\sqrt{6a^4b^4-a^8-b^8}\}$. From this equation and $x^2+2xy+2y^2=b^2$, x and y can be found.

11. $x=5,\ 3,\ -3,\ -5$; $y=3,\ 5,\ -5,\ -3$. Other solutions can be found from $xy=-\frac{79}{2}$ and $x^2+xy+y^2=49$.

12. $x+y=\sqrt[3]{a^3+3b^3}$, $xy=\frac{b^3}{\sqrt[3]{a^3+3b^3}}$.

13. $x+y=\frac{a}{\sqrt{a^2-b^2}}\sqrt{n-b^2m}$; $x-y=\frac{b}{\sqrt{a^2-b^2}}\sqrt{a^2m-n}$.

14. $x+y=\frac{a}{\sqrt{a^2+b^2}}\sqrt{n-mb^2}$, $x-y=\frac{b}{\sqrt{a^2+b^2}}\sqrt{n+ma^2}$.

15. $x=0,\ \frac{27}{8}$; $y=0,\ \frac{9}{4}$.

16. $x=\frac{a^2}{b}$, $y=\frac{b^2}{a}$.

17. $x=4,\ 0$; $y=2,\ 0$.

18. $x=9,\ 4$; $y=4,\ 9$.

19. $x=0,\ 8^{\frac{1}{4}},\ 2\left(\frac{27}{64}\right)^{\frac{1}{4}}$; $y=0,\ 2^{\frac{1}{4}},\ \left(\frac{3}{4}\right)^{\frac{1}{4}}$.

20. $x=0,\ \frac{1}{4}(1\pm\sqrt{2});$
$y=0,\ \frac{1}{4}.$

21. $x=2,\ 16;$
$y=2,\ \frac{1}{2}.$

22. $\left.\begin{array}{l} x=\ \ 8, \\ y=27. \end{array}\right\}$ The only other solution is $\left\{\begin{array}{l} x=\ \ -8, \\ y=-27. \end{array}\right.$

23. $x=3,\ \frac{1}{3};$
$y=1,\ 1.$

24. $x=0,\ \frac{3\pm\sqrt{5}}{2};$
$y=0,\ \frac{1\pm\sqrt{5}}{2}.$

25. $x=3,\ 2,\ \frac{1}{2}(5\pm\frac{1}{3}\sqrt{-309});$
$y=2,\ 3,\ \frac{1}{2}(5\mp\frac{1}{3}\sqrt{-309}).$

26. $x=\pm 2,\ \pm 2\sqrt{-1},\ \pm 5,\ \pm 5\sqrt{-1};$
$y=\pm 5,\ \pm 5\sqrt{-1},\ \pm 2,\ \pm 2\sqrt{-1}.$
$\left\{\begin{array}{l}\text{Other solutions from} \\ 29x^2+20xy+29y^2=0 \\ \text{and}\quad x^4+y^4=641.\end{array}\right.$

27. $\left.\begin{array}{l} xy=\dfrac{a^2(\sqrt{5}+1)}{8}, \\ x-y=a. \end{array}\right\}$ Solve.

28. $x=1,\ -1;$
$y=\frac{1}{3},\ -3.$

29. $\left.\begin{array}{l} x=2, \\ y=3. \end{array}\right\}$ Other solutions obtained from a cubic equation.

30. $\left.\begin{array}{l} x=n,\ -n; \\ y=m,\ -m. \end{array}\right\}$ Equations not independent.

31. $\left.\begin{array}{l} x=\sqrt{3}+\sqrt{2},\ \sqrt{3}-\sqrt{2}; \\ y=\sqrt{3}-\sqrt{2},\ \sqrt{3}+\sqrt{2}; \end{array}\right\}$ etc.

32. $x=\frac{1}{3},\ -\frac{4}{3};$
$y=\frac{1}{3}\sqrt{3},\ \frac{2}{3}\sqrt{-3}.$

33. $x=9,\ 25;$
$y=25,\ 9.$

34. $x=\dfrac{ac}{a+b},$
$y=\dfrac{bc}{a+b}.$

35. $x=-1,\ \dfrac{a(b+c)}{b};$
$y=-1,\ \dfrac{b(a+c)}{a}.$

36. $x = 1,\ 1 \pm \sqrt{-2};$
$y = 4,\ \ -2.$

37. $x = 3,\ -2;$
$y = -2,\ 3.$

38. $x = (\sqrt{a} \pm \sqrt{b})^2,$
$y = (\sqrt{a} \mp \sqrt{b})^2.$

39. $x = 5,\ 11;$
$y = \pm 4,\ \pm 4\sqrt{7}.$

40. $x = 0,\ \sqrt[3]{m+n},\ \sqrt[3]{m + \frac{n}{2}(a \mp \sqrt{a^2-4})};$

$y = 0,\ \sqrt[3]{m+n},\ \sqrt[3]{m + \frac{n}{2}(a \pm \sqrt{a^2-4})},$

where $a = \dfrac{-(m+n) \pm \sqrt{m^2 - 2mn + 5n^2}}{2n}.$

41. $\left.\begin{array}{l} x = \pm 3,\ \pm 3\sqrt{-1}; \\ y = \pm 2,\ \pm 2\sqrt{-1}. \end{array}\right\}$ Other solutions from $\dfrac{x^2}{y^2} = \dfrac{13}{5}.$

42. $x = 2, 4;$
$y = 4, 2.$

43. $x = 2, 8;$
$y = 8, 2.$

44. $x = 2\sqrt[3]{5},\ 2\sqrt[3]{2},\ 2;$
$y = 2\sqrt[3]{5},\ 4\sqrt[3]{2},\ 6.$

45. $x = a, b;$
$y = b, a.$

EXERCISE XIX. (PAGE 201.)

1. $x = 3,\ 3;$
$y = 4,\ -1;$
$z = -1,\ 4.$

2. $x = -4,\ 7,\ \dfrac{5 \pm \sqrt{137}}{2};$
$y = 7,\ -4,\ \dfrac{5 \mp \sqrt{137}}{2};$
$z = 5,\ 5,\ 3.$

3. $x = 5, 3;$
$y = 3, 5;$
$z = 7, 7.$

4. $x = 2,\ \dfrac{1}{2};$
$y = \dfrac{1}{3},\ \dfrac{1}{3};$
$z = -1,\ -4.$

5. $x = 3,\ -2,\ -1;$
$y = -2,\ -1,\ 3;$
$z = -1,\ 3,\ -2.$

6. $x = 3,\ 4,\ -5;$
$y = 4,\ -5,\ 3;$
$z = -5,\ 3,\ 4.$

7. $x = 10,\ -3,\ -2;$
$y = -3,\ -2,\ 10;$
$z = -2,\ 10,\ -3.$

8. $x = -5,\ 3,\ 1;$
$y = 1,\ -5,\ 3;$
$z = 3,\ 1,\ -5.$

9. $x = 3,\ 2,\ -1;$
$y = -1,\ 3,\ 2;$
$z = 2,\ -1,\ 3.$

10. $x = 2,\ 3,\ 4;$
$y = 3,\ 4,\ 2;$
$z = 4,\ 2,\ 3.$

11. $x = 3,\ -3;$
$y = -5,\ 5;$
$z = 8,\ -8.$

12. $x = 3,\ -5;$
$y = 1,\ -3;$
$z = -5,\ 3.$

13. $x = 1 \pm \sqrt{\dfrac{bc}{a}},$

$y = 1 \pm \sqrt{\dfrac{ca}{b}},$

$z = 1 \pm \sqrt{\dfrac{ab}{c}}.$

14. $x = \dfrac{b^2c^2 + a^2b^2 - a^2c^2}{2b^2\sqrt{b^2 - c^2}},$

$y = \dfrac{a^2c^2 + b^2c^2 - a^2b^2}{2b^2\sqrt{b^2 - c^2}},$

$z = \dfrac{2b^4 + a^2b^2 - b^2c^2 - a^2c^2}{2b^2\sqrt{b^2 - c^2}}.$

15. $x = \pm\dfrac{a(b^2 + c^2)}{2bc},$

$y = \pm\dfrac{b(a^2 + c^2)}{2ca},$

$z = \pm\dfrac{c(a^2 + b^2)}{2ab}.$

16. $x = \pm\dfrac{a^4 - b^2c^2}{\sqrt{a^6 + b^6 + c^6 - 3a^2b^2c^2}},$

$y = \pm\dfrac{b^4 - c^2a^2}{\sqrt{a^6 + b^6 + c^6 - 3a^2b^2c^2}},$

$z = \pm\dfrac{c^4 - a^2b^2}{\sqrt{a^6 + b^6 + c^6 - 3a^2b^2c^2}}.$

17. $x = \dfrac{1}{3}\{\pm\sqrt{b^2 + 2a^2} \pm \sqrt{3(a^2 - b^2)}\},\quad \dfrac{1}{3}\{\pm\ \sqrt{b^2 - a^2} \pm \sqrt{2a^2 + b^2}\};$

$y = \dfrac{1}{3}\{\pm\sqrt{b^2 + 2a^2} \mp \sqrt{3(a^2 - b^2)}\},\quad \dfrac{1}{3}\{\pm\ \sqrt{b^2 - a^2} \pm \sqrt{2a^2 + b^2}\};$

$z = \pm\dfrac{\sqrt{2a^2 + b^2}}{3},\quad \dfrac{1}{3}\{\mp 2\sqrt{b^2 - a^2} \pm \sqrt{2a^2 + b^2}\}.$

18. $x = \pm 4,\ \pm\dfrac{10}{3}\sqrt{3};$

$y = \pm 3,\ \pm\dfrac{1}{3}\sqrt{3};$

$z = \pm 2,\ \mp\dfrac{8}{3}\sqrt{3}.$

19. $x = -2,\ 2;$
$y = 4,\ -1;$
$z = -1,\ 4.$

20. $x = 4, \quad 4;$
$y = 1, \quad 3;$
$z = 6, \ -2.$

21. $x^3 = \dfrac{-a^3(R - c^3)^2}{(b^3 + c^3)(R + b^3)}$, etc.
(See Art. 207, Ex. 5.)

22. $x = 5,$
$y = -2,$
$z = 1.$

23. $x = a, 0, 0, \dfrac{(ca - b^2)(ab - c^2)}{3abc - a^3 - b^3 - c^3};$
$y = 0, b, 0,$ etc.;
$z = 0, 0, c,$ etc.

24. $x = \pm \dfrac{(b+c)^2 - (a+b)(c+a)}{2\sqrt{3abc - a^3 - b^3 - c^3}},$

$y = \pm \dfrac{(c+a)^2 - (b+c)(a+b)}{2\sqrt{3abc - a^3 - b^3 - c^3}},$

$z = \pm \dfrac{(a+b)^2 - (c+a)(b+c)}{2\sqrt{3abc - a^3 - b^3 - c^3}}.$

25. $x = \dfrac{1}{4abc}\{(bc - ca + ab)(bc + ca - ab)\}$, etc.

26. $x(ac - ab - bc) = y(ab + ac - bc) = z(ab + bc + ca).$

27. $x = \dfrac{1}{3}(2a - b - c),$

$y = \dfrac{1}{3}(2b - c - a),$

$z = \dfrac{1}{3}(2c - a - b).$

28. $x = 1\frac{1}{2}, \ 2\frac{2}{3};$
$y = 2\frac{2}{3}, \ 1\frac{1}{2};$
$z = 2, \ 2.$

29. $x = \left\{\dfrac{(a^3b^3 - c^6)(a^3 + c^3)}{(b^3 + c^3)^2}\right\}^{\frac{1}{3}},$

$y = \left\{\dfrac{(a^3b^3 - c^6)(b^3 + c^3)}{(c^3 + a^3)^2}\right\}^{\frac{1}{3}},$

$z = c^3\left\{\dfrac{(b^3 + c^3)(c^3 + a^3)}{(a^3b^3 - c^6)^2}\right\}^{\frac{1}{3}}.$

30. $x = 3, \ 4, \ 3;$
$y = 3, \ 3, \ 4;$
$z = 4, \ 3, \ 3.$

31. $x = 0, \quad \dfrac{\pm\sqrt{6} - 1}{2};$

$y = \dfrac{1 + \sqrt{-1}}{2}, \quad \dfrac{\mp\sqrt{6} - 1}{2};$

$z = \dfrac{1 - \sqrt{-1}}{2}, \quad 2.$

32. $x = 1, \quad 9;$
$y = 2, \ -6;$
$z = 4.$

33. $x = 6, \ 3\frac{1}{3};$
$y = 5, \ 6\frac{1}{3};$
$z = 3, \ 4\frac{1}{3}.$

EXERCISE XX. (PAGE 208.)

1. $z^3 + 2b^3 - 3a^2z = 0.$

2. $a^7 - 7a^5z^2 + 14a^3z^4 - 7az^6 = b^7.$

3. $a^3 + b^3 + c^3 + abc = 0.$

4. $\left(\frac{x}{a} + \frac{y}{b}\right) = \sqrt{2}.$

5. $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{(a-b)^2}.$

7. $(b+c)^{\frac{2}{3}} + (b-c)^{\frac{2}{3}} = 2a.$

8. $a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0.$

9. $\frac{1}{bc-a^2} + \frac{1}{ac-b^2} + \frac{1}{ab-c^2} = 0.$

10. $p^2(a+b+c)^2 = 3.$

11. $a^2 + b^2 + c^2 - ab - bc - ca = 0.$

13. $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 1.$

14. $(x-h)(y-k) = 0.$

15. $y = 0$ or $x = 0.$

16. $y^3 - 8 = 0$ or $x^3 - 8 = 0.$

17. $2y^2 + 1 = 0,\ y^2 - 4y + 3 = 0.$

18. $\frac{(c^2 - ab)^3}{a^6} + \frac{a^3}{b^3} = \frac{1}{8}$ or $\frac{(c^2 - ab)^3}{b^6} + \frac{b^3}{a^3} = \frac{1}{8}.$

19. $(ac_1 - a_1c)^2 = (b_1c - bc_1)(a_1b - ab_1).$

20. $4a^4b^8 - 2a^4c^8 - a^8b^4 - b^{12} = 0.$

21. $\frac{a^4}{b^4} = \frac{n-m}{n+m}.$

22. $p^2 = b^2(a^2 + m^2).$

23. $c^2 + a^2 - b^2 = 0.$

24. $4m^2ca + 4n^2ab + p^2a^2 = 4mnap + 4a^2bc.$

25. $(ab_1 - a_1b)^2 = (b_1c - bc_1)^2 + (ac_1 - a_1c)^2.$

27. By squaring the equations and adding we get

$$(a^6 + b^6) + 2(a^4 + b^4)x + (a^2 + b^2)x^2 = y^2 + z^2.$$

But $\qquad a^2 + b^2 = 1, \quad \therefore\ a^2b^2 = \frac{(1+x)^2 - y^2 - z^2}{4x+3} \qquad (1)$

Also, by adding and subtracting the equations and multiplying together,

$$(1+x)^2 - a^2b^2 = \frac{y^2 - z^2}{a^2 - b^2}. \qquad (2)$$

From (1) and (2) a^2 and b^2 can be found, and their values substituted in $a^2 + b^2 = 1.$

28. $\frac{x^2}{m^2} + \frac{y^2}{n^2} + \frac{z^2}{c^2} = 1.$

32. $\frac{x^2}{y(x-z) - y^2} = \frac{z(y-x)}{x^2} - \frac{y^2}{[illegible]}.$

33. $a^3 - 3ab^2 + 2c^3 = 0$.
34. $a^3 - 3ab^2 + 2c^3 - 6d^3 = 0$.
35. $a + b + c = 0$.
36. $c^3 = a^3(a + b + 2)$.
37. $a^2 + b^2 + c^2 + 2abc = 1$.
38. $a^3 + b^3 + c^3 + abc = 0$.
39. $(a + b + c - 1)^2 = 4abc$.
40. $a^3 + b^3 + c^3 - 3abc = d^2$.
41. $a^2x_1^2 + b^2y_1^2 + c^2z_1^2 = \frac{a^4x_1^2}{a^2 + k^2} + \frac{b^4y_1^2}{b^2 + k^2} + \frac{c^4z_1^2}{c^2 + k^2}$.
42. $ab + bc + ca + 2abc = 1$.
43. $(a^2 + b^2 + c^2)^3 + 8(ab + bc + ca)^3 = 0$.
44. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = z^{\frac{1}{2}}$
45. $5(a^3 - b^3)(2a^3 + b^3) = 9a(a^5 - c^5)$.
46. $\left(\frac{c^2 + a^2}{ac}\right)^{\frac{2}{5}} - \left(\frac{c^2 - a^2}{ac}\right)^{\frac{2}{5}} = 1$.
47. $ab = c + 1$.
48. $(a + b)^{\frac{2}{3}} - (a - b)^{\frac{2}{3}} = (8c)^{\frac{2}{3}}$.

EXERCISE XXI. (PAGE 217.)

1. $(x - 2y - 2a)(2x - y + 3a)$.
2. $(ac - 2d)^2 = (a^2 - 4b)(c^2 - 4e)$.
9. Between $\frac{3}{2}$ and $-\frac{7}{2}$.
17. Between 2 and -12.
21. $m = -2$.
22. $m = \pm 7$.
24. $(ln_1 - l_1n)^2 = (lm_1 - l_1m)(mn_1 - m_1n)$.
26. $(aa_1 - bb_1)^2 + 4(ha_1 + h_1b)(hb_1 + h_1a) = 0$.
29. $(ac_1 - a_1c)^2 = (ab_1 - a_1b)(bc_1 - b_1c)$.
31. 576.

EXERCISE XXII. (PAGE 223.)

1. $x = \frac{1 \pm \sqrt{-3}}{2}$, 3, $-\frac{2}{3}$.
2. $x = 2 \pm \sqrt{3}$, $-\frac{3}{2}$, $-\frac{1}{3}$.
3. $x = -1 \pm \sqrt{-1}$, $-1 \pm \sqrt{2}$.
4. $x = \pm\sqrt{3}$, $1 \pm 2\sqrt{-1}$, -1.
5. (1) $x^4 - 2x^2 + 25 = 0$, (2) $x^4 - 8x^2 + 36 = 0$,
(3) $x^4 - 2x^2 + 9 = 0$, (4) $x^4 - 10x^2 + 1 = 0$,
6. $x^4 - 10x^3 - 19x^2 + 480x - 1392 = 0$.
7. $x^8 - 16x^6 + 88x^4 + 192x^2 + 144 = 0$,

EXERCISE XXIII. (PAGE 230.)

1. (1) $\dfrac{q}{-r}$, (2) $-p^3+3pq-3r$, (3) $\dfrac{pq-3r}{r}$.

2. (1) 3, (2) -2.

3. (1) $x^3-(p^2-2q)x^2+(q^2-2pr)x-r^2=0$,
(2) $x^3-qx^2+prx-r^2=0$.

4. 2, 2, 1.

5. $\dfrac{3}{4}$, -2, $-\dfrac{7}{16}$.

6. $(27r+2p^3-9pq)^2=4(p^2-3q)^3$.

8. (1) $x^3+2px^2+x(p^2+q)+pq-r=0$,
(2) $x^3-x^2(q^2-2pr)+x(p^2r^2-2qr^2)-r^4=0$.

9. $x^3-2qx^2+q^3x+r^2=0$.

10. $q^3=-8r^2$, $x=\dfrac{3}{2}$, $-\dfrac{3\pm3\sqrt{5}}{4}$.

11. $m^2=3n$.

12. $qc-q^2a-pq(b-pa)=e$,
$qd-pe-q^2(b-pa)=0$.

13. $4q=4(m+1)+p^2$.

14. $(B^2-CA)(D^2-AF)=(BD-EA)^2$.

15. $(2x^2+x+2)^2-5x^2$.

16. (1) $4bcd=8ad^2+c^3$, (2) $b^3+8a^2d=4abc$; $x=1$, $\dfrac{1}{2}$, -1.

21. (1) $\dfrac{3q}{p}$, (2) $\dfrac{3(3q-2pr)}{p^2}$, (3) $\dfrac{-27q(q^2-pr)}{p^3}$.

23. 0.

24. 1.

27. $a=1$, $b=11$, $c=11$, $d=1$, $e=0$.

28. $\dfrac{1}{2}+\dfrac{3}{4}x+\dfrac{9}{8}x^2+\ldots$
$\dfrac{1}{2}-\dfrac{1}{4}x+\dfrac{3}{8}x^2-\dfrac{9}{16}x^3+\ldots$
$1-x^2-x^3+x^5+x^6-x^8-x^9+\ldots$

30. $x=a+2c$,
$y=b+3c$.

31. $\dfrac{n(n+1)(n+2)}{3}$.

32. $\left\{\dfrac{n(n+1)}{2}\right\}^2$.

33. n^2.

EXERCISE XXIV. (PAGE 239.)

1. 504.

2. 25.

3. 300, 1190.

4. 20, 36.

5. 6, 48.

6. 24.

7. 3628800, 39916800, 239500800.

8. 60, 325.

9. 1555.

10. 8.

11. 13.

12. 720, 518400, 55440.

13. 34650, 151200, 121080960.

14. 114.

15. 3000.

16. 60, 12.

17. 35.

18. 30240.

19. 90720, 70560, 17640.

20. $\underline{|13}$, $\dfrac{\underline{|20}}{\underline{|8}}$, $\underline{|13}\,\underline{|8}$.

21. 122880. 22. 494020. 23. 27, 10, 18. 24. 80640.
25. 40320, 5040, 2520. 26. 24. 27. 3628800.
28. 282240. 29. 298598400, 8233505280. 30. 24, 24, 72.
31. 362880, 2903040. 32. $|\underline{m} \,.\, |\underline{m-1}$. 33. 81126230400.

EXERCISE XXV. (Page 250.)

1. 210, 84. 2. 38760, 3060, 8568. 3. 20.
4. 46558512, 5587021440. 5. 630. 6. 51.
7. 1023, 512. 8. 127, \$762.24, 171. 9. 791.
10. 24, 30 (including unity and the given number).
11. 163, 3393, 3386880. 12. 576, 821; 46866, 314695.
13. 36 or 39 cents. 14. 5, 2. 15. 8 or 9, 2 or 3, 243100.
16. $\dfrac{|\underline{22}}{|\underline{11}\,|\underline{11}}$, $2^{22}-1$, $(2^{17}-18)(2^5-1)$. 17. 2^n. 18. 12.
19. 8. 20. 12, 4. 21. $\dfrac{|\underline{m}\,|\underline{m-1}}{|\underline{m-n}}$.
22. 390625. 23. 5. 24. $(p+1)(q+1)2^{n-q-p}-1$.
25. 21, 56, $\dfrac{|\underline{n+5}}{|\underline{5}\,|\underline{n}}$. 26. 1820. 27. 56.
28. 8204716800. 29. $\dfrac{|\underline{m}}{|\underline{r}\,|\underline{m-r}} \cdot \dfrac{|\underline{n}}{|\underline{s}\,|\underline{n-s}} \cdot |\underline{r+s}$.
30. 244. 31. 19. 32. $\dfrac{n(n-1)}{|\underline{2}}$, $\dfrac{n(n-1)(n-2)}{|\underline{3}}$.
33. $\dfrac{n(n-1)}{|\underline{2}}$, $\dfrac{n(n-1)(n-2)}{|\underline{3}}$, $\dfrac{n(n-1)(n-2)(n^3-13n+20)}{48}$,

$$\frac{n(n-1)(n-2)(n-3)}{8}.$$

34. $\dfrac{n(n-1)}{|\underline{2}} - \dfrac{p(p-1)}{|\underline{2}} + 1$, $\dfrac{|\underline{n-p}}{|\underline{3}\,|\underline{n-p-3}} + \dfrac{p\,|\underline{n-p}}{|\underline{2}\,|\underline{n-p-2}}$

$$+\frac{p(p-1)(n-p)}{|\underline{2}}.$$

35. $\dfrac{1}{2}\,|\underline{n-1}$.

36. $\dfrac{m(m-1)(n+p)}{2}+\dfrac{n(n-1)(p+m)}{2}+\dfrac{p(p-1)(m+n)}{2}+mnp.$

37. 204, $\dfrac{n(n+1)(2n+1)}{6}$. 38. 6084, $\dfrac{n^2(n+1)^2}{4}$.

EXERCISE XXVI. (Page 259.)

1. 20, 10. 2. 5775, 34650. 3. $\dfrac{\underline{|26}}{(\underline{|6})^4\,\underline{|2}\,\underline{|4}}$.

4. $\dfrac{\underline{|nr}}{(\underline{|r})^n}$. 5. 1663200. 6. $\dfrac{\underline{|23}}{\underline{|3}}$.

7. 1001. 8. $\dfrac{\underline{|100}}{\underline{|95}}$, 10^{10}. 9. 75600.

10. $\dfrac{\underline{|26}\,\underline{|25}}{\underline{|5}\,\underline{|20}}$. 11. $\dfrac{\underline{|99}}{\underline{|9}\,\underline{|90}}$. 12. 969, 1771.

13. $\dfrac{\underline{|r+n-1}}{\underline{|n}\,\underline{|r-1}}$, $\dfrac{\underline{|n+r}}{\underline{|n}\,\underline{|r}}$. 14. 7. 15. 2^{2n}.

16. $\dfrac{n}{2}\left(\dfrac{n}{2}-1\right)$ or $\left(\dfrac{n-1}{2}\right)^2$, according as n is *even* or *odd*.

17. $(n+1)^2$. 18. 30786. 19. 576.

21. 46376. 22. $\dfrac{\underline{|pq+r}}{(q+1)^r(\underline{|q})^p}$. 23. $\dfrac{\underline{|p+1}}{\underline{|n}\,\underline{|p-n+1}}$.

24. 690. 25. $3n^2+3n+1$. 26. 209952. 27. 2815

EXERCISE XXVII. (Page 265.)

9. $1-\dfrac{1}{2^n}$. 10. $\dfrac{2n(n+1)(2n+1)}{3}$. 11. $\dfrac{n}{n+1}$.

15. $P(n-1)\left\{\left(\dfrac{n}{n-1}\right)^n+(n-1)\right\}$. 16. $a^2+4ac=b^2$.

EXERCISE XXVIII. (PAGE 275.)

1. $x^3 + x^2(a + b - c) + x(ab - bc - ca) - abc$.

2. $x^4 - 10x^3 + 7x^2 + 162x - 360$. 3. $x^4 - 225x^2 + 1620x - 2916$.

4. $x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + a^5$.

5. $64a^6 + 576a^5b + 2160a^4b^2 + 4320a^3b^3 + 4860a^2b^4 + 2916ab^5 + 729b^6$.

6. $16a^4 - 32a^3y + 24a^2y^2 - 8ay^3 + y^4$.

7. $1 - 12x + 60x^2 - 160x^3 + 240x^4 - 192x^5 + 64x^6$.

8. $x^6 + 12x^4 + 60x^2 + 160 + \dfrac{240}{x^2} + \dfrac{192}{x^4} + \dfrac{64}{x^6}$.

9. $x^8 + 4x^4 + 6 + \dfrac{4}{x^4} + \dfrac{1}{x^8}$. 10. $252x^6y^2$. 11. $-20000x^3$.

12. $-\dfrac{\underline{|25}}{\underline{|19}\,\underline{|6}}a^6b^{19}$. 13. $-\dfrac{\underline{|40}}{\underline{|34}\,\underline{|6}}4^6x^6a^{34}$. 14. $\dfrac{\underline{|10}}{\underline{|5}\,\underline{|5}}x^5$.

15. $\dfrac{\underline{|12}}{\underline{|6}\,\underline{|6}}2^6x^6y^6$. 16. $\dfrac{\underline{|12}}{\underline{|6}\,\underline{|6}}x^6$. 17. $(-1)^r\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}3^r.x^{n-r}y^r$.

18. $\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}x^{2n-2r}y^{3r}$. 19. $(-1)^r\dfrac{\underline{|16}}{\underline{|r}\,\underline{|16-r}}x^{32-2r}y^{3r}$.

22. $\dfrac{\underline{|n}}{\underline{|\tfrac{1}{2}(n+r)}\,\underline{|\tfrac{1}{2}(n-r)}}$. 23. $(-1)^n\dfrac{\underline{|2n}}{\underline{|n}\,\underline{|n}}$. 24. $n = 7$.

25. $n = 7$. 26. $2x^4 + 12x^2y^2 - 12x^2 + 2y^4 - 4y^2 + 2$.

27. $16(4m^4 - 1)\sqrt{m^4 - 1}$. 29. 1. 30. $\dfrac{5}{192}$.

31. $\dfrac{\underline{|2n}}{\underline{|\tfrac{2n}{3}}\,\underline{|\tfrac{4n}{3}}}$ (when n is a multiple of 3).

32. $\dfrac{\underline{|n}}{\underline{|\tfrac{(2n-r)}{5}}\,\underline{|\tfrac{3n+r}{5}}}$. 33. $\dfrac{\underline{|2n}}{\underline{|\tfrac{4n+m}{5}}\,\underline{|\tfrac{6n-m}{5}}}$. 34. $2^n + n\,.\,2^{n-1}$.

EXERCISE XXIX. (PAGE 281.)

1. 13^{th} term. 2. 5^{th} and 6^{th} terms. 3. 4^{th} term.

4. 5^{th} term. 5. $\frac{4}{3}$. 6. 540.

7. 108864. 8. 512. 9. $r = 14$.

10. $x = 3$, $n = 5$, $a = 2$. 11. $n = 2r + 8$. 12. $2r = n$.

EXERCISE XXX. (PAGE 293.)

1. $1 + \frac{3}{4}x - \frac{3}{32}x^2 + \frac{5}{128}x^3 - \dots.$ 2. $1 - \frac{2}{3}x - \frac{1}{9}x^2 - \frac{4}{81}x^3 - \dots.$

3. $1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \dots.$

4. $a^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}bx - \frac{1}{8}a^{-\frac{3}{2}}b^2x^2 + \frac{1}{16}a^{-\frac{5}{2}}b^3x^3 - \dots.$

5. $2^{\frac{2}{5}} - 2^{\frac{2}{5}}x - 2^{-\frac{3}{5}}\,3x^2 - \dots$ 6. $1 - 3x + 6x^2 - 10x^3 + \dots.$

7. $1 + 6x + 21x^2 + 56x^3 + \dots.$ 8. $\frac{1}{8} + \frac{3}{16}x + \frac{3}{16}x^3 + \frac{5}{32}x^4 + \dots$

9. $\frac{1}{3^{\frac{2}{3}}}(1 + \frac{4}{3^2}x + \frac{20}{3^4}x^2 + \frac{320}{3^8}x^3 + \dots).$

10. $1 - 3x^2 + 6x^4 - 10x^6 + \dots.$

11. $a^2 - \frac{2}{3}\cdot\frac{x^3}{a} - \frac{1}{9}\cdot\frac{x^6}{a^4} - \dots. \frac{2.1.4\dots(3r-5)}{3^r\lfloor r}\cdot\frac{x^{3r}}{a^{3r-2}} - \dots.$

12. $1 - \frac{3}{2}x + \frac{27}{8}x^2 - \frac{135}{16}x^3 - \dots.$

13. $\frac{1}{a} + \frac{x}{a^{n+1}} + \frac{x^2(n+1)}{2a^{2n+1}} + \dots.$

$$\frac{(n+1)(2n+1)\dots\{(r-1)n+1\}}{\lfloor r}\cdot\frac{x^r}{a^{nr+1}} + \dots.$$

14. $\frac{-63}{8}x^5.$ 15. $\frac{-9}{16}y^{11}.$ 16. $\frac{35}{3}x^{12}.$

17. $\frac{-225}{2^6\lfloor 6}a^{-\frac{7}{2}}x^6.$ 18. $\frac{-(7.4.2.5.8.11.14.17)2^9}{3^{\frac{17}{2}}\lfloor 9}.b^9.$

19. $(-1)^r.\frac{4.7.10\dots(3r+1)}{3^r\lfloor r}x^r.$ 20. $\frac{11.8.5\dots(14-3r)}{3^r\lfloor r}x^r.$

21. $(-1)^r . \dfrac{3.4.5\ldots(r+2)}{\lfloor r}(-2)^r . y^r = (r+1)(r+2)2^{r-1}y^r$.

22. $3^r\left\{\dfrac{(r+1)(r+2)(r+3)}{\lfloor 3}\right\} m^{-(r+4)}n^r$.

23. $(-1)^r \dfrac{1.3.5\ldots(2r-1)}{\lfloor r} x^{\frac{-(2r+1)}{2}} y^r$.

24. $-2^{21}\left\{\dfrac{13.11.9.7.5.3.3.5.7}{\lfloor 11}\right\} x^{11}$.

25. $\dfrac{11.9.7.5.3}{\lfloor 5} 3^{24} \times 2^{15} x^5$. 26. $\dfrac{3.5.9.13.17.21}{2^{\frac{53}{2}} \lfloor 7} . 3^7 . x^7$.

27. $\dfrac{15}{32}x^2$, $\dfrac{3.5.7.9.11}{2^{10} \lfloor 5} x^5$. 28. $\dfrac{(r+1)(r+2)(r+3)(r+4)}{\lfloor 4} x^r$.

29. $x^4 + 4x^3 + 10x^2 + 20x \ldots$ or $x^8(1 + 4x + 10x^2 + \ldots)$.

30. $\dfrac{1.3.5.7.9\ldots21}{\lfloor 11} 2^{11} . x^{11}$.

EXERCISE XXXI. (Page 297.)

1. The 4^{th} or 5^{th} terms. 2. The 23^{rd} term.
3. The 39^{th} term. 4. The 12^{th} term.
5. The 5^{th} term. 6. The 7^{th} term.
7. The 3^{rd} term. 8. The 9^{th} term. 9. The 8^{th} term.

EXERCISE XXXII. (Page 308.)

1. -19. 2. $(-1)^{m-1}$. 3. $2m^2 + 3m + 2$.

6. (1) $9{\cdot}99666\ldots$, (2) $10.00666\ldots$,
(3) $6{\cdot}99927 + \ldots$, (4) $5{\cdot}00128$.

8. (1) $1 - \dfrac{5x}{8}$, (2) $\dfrac{2}{3}\left(1 + \dfrac{x}{24}\right)$. 15. $\dfrac{-245}{8}$.

16. Coefficient of x^{3r} is $3^{3r} . 2^{-3r-2} . a^{-3r}$; coefficient of x^{3r+1} is $-3^{3r+1} . 2^{-3r-3} . a^{-3r-3}$, and of x^{3r+2} is 0.

18. $\dfrac{(r+1)(r+2)(r+3)(r+4)(r+5)}{30}$. 23. $3\sqrt{3} - 2$. 53. 462.

55. $mn + \dfrac{1}{2}m(m+1)$.

EXERCISE XXXIII. (PAGE 318.)

1. \$1080, $12\frac{1}{2}$ per cent. 2. \$750. 5. \$500. 6. $\dfrac{B-A}{A}$.

7. \$252.13, \$330.92, \$416.95. 8. $57\frac{1}{7}$, 32 nearly.

9. \$496.97. 10. 48.2 years nearly.

11. $\dfrac{\log 2}{\log(mn-m+n)-\log mn}$. 12. 17.67.

13. \$1.00, \$23912 $(10)^{36}$.

EXERCISE XXXIV. (PAGE 323.)

1. \$374.11. 2. \$3137.14. 3. $4\frac{1}{2}$ per cent. 4. $28\frac{4}{7}$.

5. \$3115.55. 6. \$3385.20. 7. \$802.42. 8. \$2199.95.

9. \$7360.08, \$6404.74, \$2901.83. 10. $3\frac{1}{5}$.

12. $$\frac{100\,(1.06)^{10}-490}{(1.06)^{\frac14}-1}+10\left\{\frac{(1.06)^{10}-(1.06)^{\frac14}}{\{(1.06^{\frac14}-1\}^2}\right\}-75\left\{\frac{(1.05)^{40}-(1.06)^{10}}{1.05-(1.06)^{\frac14}}\right\}$$

13. 10.74. 14. £1308 12s. $4\frac{1}{2}$d. 15. \$4200.

16. $\dfrac{\log(m-1)}{\log(1+r)}$. 18. $\dfrac{n^2(n+1)}{4}\{2+(n-1)r\}$.

19. $\dfrac{P^2}{P-Q}\left\{\left(\dfrac{P}{Q}\right)^{-\frac{1}{n}}-1\right\}$ 20. $\dfrac{A}{r}\left\{\dfrac{n(n+1)}{2}-\dfrac{(1+r)^n-nr-1}{(1+r)^{n-1}r^2}\right\}$

21. $\dfrac{\log n-\log(n-1)}{\log(1+r)}$, $\dfrac{\log(n-1)-\log(n-2)}{\log(1+r)}$ $\ldots$ $\dfrac{\log 2}{\log(1+r)}$.

MISCELLANEOUS EXAMPLES.

EXERCISE XXXV. (PAGE 327.)

1. $1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9$. 4. $x-1$.

5. $\dfrac{a^2}{c}+\dfrac{b^2}{d}=1,\ x=\dfrac{a}{c},\ y=\dfrac{b}{d}$. 7. $(a+b)^7$.

8. $x=n(bc-a^2),\ y=n(ca-b^2),\ z=n(ab-c^2)$.

9. $\dfrac{3(y^2-zx)}{x^3+y^3+z^3-3xyz},\ \dfrac{3(z^2-xy)}{x^3+y^3+z^3-3xyz},\ 27xyz,$
$\dfrac{27(x^2-yz)(y^2-zx)(z^2-xy)}{(x^3+y^3+z^3-3xyz)^3}$

10. $\dfrac{r-2}{3},\ r-1,\ \dfrac{2r-1}{3}$; scales of 5, 8, 11, etc.

19. $\dfrac{(x-p)(x-q)}{(x+a)(x+b)(x+c)}$. 20. 999, 2220. 21. $\dfrac{d}{2d+l},\ l:2d+l$.

22. $\dfrac{1}{x-y}\left\{\dfrac{x(1-x^n)}{1-x}-\dfrac{y(1-y^n)}{1-y}\right\}$. 23. $\dfrac{x^{2n}-y^{2n}}{(x+y)x^{2n-2}}$.

24. $\dfrac{18}{5}(6^n-1),\ \dfrac{x^2(x^{2n}-1)}{x^2-1}-\dfrac{n(n+1)(2n+1)}{6},\ \dfrac{(xy)^{n+1}-(xy)^{-n}}{xy-1},$
$\dfrac{3}{5}(6^n)-2^n+\dfrac{2}{5}$.

25. $a:b:c=1:\sqrt[4]{2}:\sqrt{1+\sqrt{2}}$. 26. $\dfrac{2(a+b)}{\left\{\left(\dfrac{a}{b}\right)^{\frac14}+\left(\dfrac{b}{a}\right)^{\frac14}\right\}^2}$.

28. $2^6\,\underline{|6},\quad \underline{|12}-6\,.\,2\,\underline{|11}+\dfrac{6\,.\,5}{1\,.\,2}\,.\,2^2\,\underline{|10}-\dfrac{6\,.\,5\,.\,4}{1\,.\,2\,.\,3}\,.\,2^3\,\underline{|9}+\text{etc.}$

29. 505. 30. $\dfrac{1}{9}(10^n-1)$. 31. 5455. 32. 190800.

38. $\dfrac{1\,.\,3\,.\,5\,\ldots\ldots\,2r-1}{\underline{|r}}(2x)^r,\ \dfrac{1\,.\,6\,.\,11\,\ldots\ldots\,5r-4}{5^r\,\underline{|r}}\cdot\dfrac{x^{5r}}{a^{5r+1}}$.

39. $\dfrac{2^{n-2}}{3}(n+1)(n+2)(5n+6)$. 40. $\dfrac{n(n+1)}{60}(6n^3+9n^2+11n+4)$.

44. $\dfrac{x}{12}$. 45. $x=a^2(b+c)+b^2(c+a)+c^2(a+b)$.

46. $x = y = z = \frac{1}{3}$.

47. $\frac{8}{6}$, $\frac{6}{4}$.

48. 10 ft., 15 ft.

49. $x = \quad 3, \; -1, \; \frac{1 \pm \sqrt{-3}}{2}$;

$y = \quad 5, \quad 1, \; \frac{1 \mp \sqrt{-3}}{2}$;

$z = -7, \quad 1, \quad 0$.

50. $(a + \sqrt{a^2 - 1})(b + \sqrt{b^2 - 1})(c + \sqrt{c^2 - 1}) = 1$.

51. $x = (a + b - c)(a - b + c)$,
$y = (b + c - a)(b - c + a)$,
$z = (c + a - b)(c - a + b)$.

56. $x + \frac{1}{x} = -4 \pm \sqrt{6}$.

60. $\frac{a^2}{l} + \frac{b^2}{m} + \frac{c^2}{n} = 0$.

61. $\frac{1}{5}\left(1 - \frac{1}{5n+1}\right)$, $\frac{1}{5}$.

76. $rx^3 - q(1 + r)x^2 - (1 + r)^3 = 0$.

84. $(ax + by + cz)(bx + cy + az)(cx + ay + bz)$.

88. $\frac{\lfloor n - r}{\lfloor r \; \lfloor n - 2r}$.

90. $\frac{x^n(4x^3 + 12px^2 + 12qx + 4r)}{x^4 + 4px^3 + 6qx^2 + 4rx + t}$.

91. m^{th} term $= 2m^3 - m^2 - 2m + 1$;

$$\text{sum} = 2\left\{\frac{n(n+1)}{2}\right\}^2 - \frac{n(n+1)(2n+1)}{6} - (n)(n+1) + n.$$

94. (1) $x = a$ and $ax^2 + ax - 1 = 0$: (2) $x = 2, \; \frac{1}{2}, \; 3, \; \frac{1}{3}$.

97. $\frac{a_1(a_1^{2n} - 1)}{a_1^2 - 1}$.

99. $-1 + \sqrt{-1}, \; -1 - \sqrt{-1}, \; -3, \; 1$.

102. 480.

103. $1 - \frac{6}{(n+1)(n+2)(2n+3)}$.

108. $= \frac{-p^4 + 4p^2q - 8pr}{4q - p^2}$.

113. 380.

118. $= \lfloor n \left\{\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \ldots + \frac{(-1)^n}{\lfloor n}\right\}$.

APPENDIX.

The following are the ordinary proofs for Permutations, Combinations and Binomial Theorem:

1. *To find the number of permutations of n things taken r at a time.*

Denote the n things by the letters a, b, c, d . . . . , and the required number by the symbol $(n)_r$.

We can form these $(n)_r$ permutations into n classes—(1) those in which a stands first, (2) those in which b stands first, and so on; and $(n)_r$ is the sum of the numbers of all these classes.

Now, every permutation of the first class can be formed by placing a before one of the permutations of the $(n-1)$ things, b, c, d . . . . , taken $r-1$ at a time; and every one of these latter permutations give a different permutation of the first class. Hence the number in this class $=(n-1)_{r-1}$. Similarly it can be shown that $(n-1)_{r-1}$ is the number in each of the n classes; $\therefore$ the sum of the numbers in all these classes $=n(n-1)_{r-1}$.

Hence $$(n)_r = n(n-1)_{r-1}$$

and $$(n-1)_{r-1} = (n-1)(n-2)_{r-2},$$

$$\text{etc.} \quad = \quad \text{etc.},$$

$$(n-r+2)_2 = (n-r+2)(n-r+1)_1.$$

$\therefore$ multiplying, $$(n)_r = n(n-1)\ldots.(n-r+2)(n-r+1)_1;$$

but $$(n-r+1)_1 = n-r+1.$$

$$\therefore\ (n)_r = n(n-1)\ldots.(n-r+1).$$

2. *To find the number of combinations of n things taken r at a time.*

Denote the n things by the letters a, b, c, d ...., and the required number by $(n)_r$.

We can form all such combinations into n classes—(1) those in which a stands first, (2) those in which b stands first, and so on; and the sum of the numbers in all these classes is $r(n)_r$. For every combination occurs r times, viz., once in each class in which any one of its component things stands first.

For instance, when $r = 4$ the combination $abcd$ occurs in each of the first four classes.

Now, every combination of the first class can be formed by placing a before one of the combinations of the $(n-1)$ things, b, c, d ...., taken $r-1$ at a time; and every one of these latter combinations gives a different combination of the first class. Hence the number of this class $= (n-1)_{r-1}$. Similarly we can show that $(n-1)_{r-1}$ is the number in each of the n classes; $\therefore$ the sum of the numbers in all these classes $= n(n-1)_{r-1}$.

Hence $$r(n)_r = n(n-1)_{r-1};$$

$$\therefore\ (n)_r = \frac{n}{r}(n-1)_{r-1},$$

and $$(n-1)_{r-1} = \frac{n-1}{r-1}(n-2)_{r-2},$$

$$\text{etc.} \quad = \quad \text{etc.},$$

$$(n-r+2)_2 = \frac{n-r+2}{2}(n-r+1)_1.$$

$\therefore$ multiplying, $$(n)_r = \frac{n(n-1)\ldots\ldots(n-r+2)}{\underline{|r}}(n-r+1)_1;$$

but $$(n-r+1)_1 = n-r+1;$$

$$\therefore\ (n)_r = \frac{n(n-1)\ldots\ldots(n-r+2)(n-r+1)}{\underline{|r}}.$$

3. *To prove the Binomial Theorem when the index is a* POSITIVE INTEGER.

We shall find, by actual multiplication, that

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc.$$

Assume this law of formation to hold for $n-1$ factors, so that

$$(x+a_1)(x+a_2)\ldots\ldots(x+a_{n-1})=x^{n-1}+p_1x^{n-2}+p_2x^{n-3}+\text{etc.}+p_{n-1}$$

where
$$p_1=a_1+a_2+a_3+\text{etc.},$$
$$p_2=a_1a_2+a_1a_3+a_2a_3+\text{etc., etc.}=\text{etc.},$$
$$p_{n-1}=a_1a_2a_3\ldots\ldots a^{n-1}.$$

Then, multiplying by another factor, $x+a_n$, we have

$$(x+a_1)(x+a_2)\ldots\ldots(x+a_n)$$
$$=x^n+p_1x^{n-1}+p_2x^{n-2}+\text{etc.}+p_{n-1}x$$
$$+a_nx^{n-1}+p_1a_nx^{n-2}+\text{etc.}+p_{n-2}a_nx+p_{n-1}a_n$$
$$=x^n+q_1x^{n-1}+q_2x^{n-2}+\text{etc.}+q_{n-1}x+q^n$$

where

$$q_1=p_1+a_n \quad =a_1+a_2+a_3+\text{etc.}+a_n,$$
$$q_2=p_2+p_1a_n=a_1a_2+a_1a_3+a_2a_3+\text{etc.}+a_1a_n+a_2a_n+\text{etc., etc.}=\text{etc.},$$
$$q_n=p_{n-1}a_n \quad =a_1a_2a_3\ldots\ldots a_{n-1}a_n;$$

that is, if the law holds for the product of $n-1$ factors, it holds also for that of n factors. But we have seen above that it *does* holds for *three* factors, therefore for *four*, and therefore for *five*, and so on; that is, it holds generally, when n is a positive integer.

Now, it is easily seen that the terms in q_1, q_2, q_3, etc., are the different combinations of the n letters, a_1, a_2, a_3, etc., a_n, taken *one*, *two*, *three*, etc., together; and, consequently, the number of terms in q_1 is C_1, in q_2 in C_2, etc., where C_1, $C_2\ldots C_n$, represent

the combinations 1, 2, 3 .... together. Let us put a for each of a_1, a_2, etc.; then the first side becomes $(x+a)^n$, and *each* of the terms in q_1, q_2, q_3, etc., becomes a, a^2, a^3, etc., respectively; and therefore we have

$$(x+a)^n = x^n + C_1 a x^{n-1} + C_2 a^2 x^{n-2} + \text{etc.}$$

$$= x^n + \frac{n}{1} a x^{n-1} + \frac{n(n-1)}{1 \cdot 2} a^2 x^{n-2} + \text{etc.}$$

And, of course, it will follow in like manner that

$$(a+x)^n = a^n + C_1 x a^{n-1} + C_2 x^2 a^{n-2} + \text{etc.}$$

$$= a^n + C_1 a^{n-1} x + C_2 a^{n-2} x^2 + \text{etc.}$$